Asset Pricing under Asymmetric Information
Rational Expectations Equilibrium

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A Classification of Market Microstructure Models

- Simultaneous submission of demand schedules
  - Competitive rational expectation models
  - Strategic share auctions
- Sequential move models
  - Screening models:
    uninformed market maker submits a supply schedule first
    - Static
      - ◇ uniform price setting
      - ◇ limit order book analysis
    - Dynamic sequential trade models with multiple trading rounds
  - Signalling models:
    informed traders move first, market maker second

Overview

- Competitive REE (Examples)
  - Preliminaries
    - LRT (HARA) utility functions in general
    - CARA Gaussian Setup
      - ◇ Certainty equivalence
      - ◇ Recall Projection Theorem/Updating
  - REE (Grossman 1976)
  - Noisy REE (Hellwig 1980)
- Allocative versus Informational Efficiency
- Endogenous Information Acquisition

Utility functions and Risk aversion

- Utility functions: $U(W)$
- Risk tolerance, $1/\rho =$ reciprocal of the Arrow-Pratt measure of absolute risk aversion

$$\rho(W) := -\frac{\partial^2 U/\partial W^2}{\partial U/\partial W}$$

- Risk tolerance is linear in W if

$$\frac{1}{\rho} = \alpha + \beta W$$

- Also called hyperbolic absolute risk aversion (HARA) utility functions

LRT(HARA)-Utility Functions

Class	**Parameters**	$U(W) =$
expo/CARA	$\beta = 0, \alpha = 1/\rho$	$-\exp\{-\rho W\}$
gen. power	$\beta \neq 1$	$\frac{1}{\beta-1}(\alpha + \beta W)^{(\beta-1)/\beta}$
a) quadratic	$\beta = -1, \alpha > W$	$-(\alpha - W)^2$
b) log	$\beta = +1$	$ln(\alpha + W)$
c) power/CRRA	$\alpha = 0, \beta \neq 1, -1$	$\frac{1}{\beta-1}(\beta W)^{(\beta-1)/\beta}$

Certainty Equivalent in CARA-Gaussian Setup

$$U(W) = -\exp(-\rho W), \text{ hence } \rho = -\frac{\partial^2 U(W)/\partial(W)^2}{\partial U(W)/\partial W}$$

$$E[U(W) \mid \cdot] = \int_{-\infty}^{+\infty} -\exp(-\rho W) f(W|\cdot) \mathrm{d}W$$

$$\text{where: } f(W|\cdot) = \frac{1}{\sqrt{2\pi\sigma_W^2}} \exp\left[-\frac{(W-\mu_W)^2}{2\sigma_W^2}\right]$$

Substituting it in:

$$E[U(W) \mid \cdot] = \frac{1}{\sqrt{2\pi\sigma_W^2}} \int_{-\infty}^{+\infty} -\exp\left(-\frac{\rho z}{2\sigma_W^2}\right) \mathrm{d}W$$

$$\text{where } z = (W-\mu_W)^2 + 2\rho\sigma_W^2 W$$

Certainty Equivalent in CARA-Gaussian Setup

Completing square:

$$z = (W - \mu_W + \rho\sigma_W^2)^2 + 2\rho\left(\mu_W - \frac{1}{2}\rho\sigma_W^2\right)\sigma_W^2$$

Hence, $E[U(W) \mid \cdot] = -\exp[-\rho(\mu_W - \frac{1}{2}\rho\sigma_W^2)] \times$

$$\times \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_W^2}} \exp\left(-\frac{(W - \mu_W + \rho\sigma_W^2)^2}{2\sigma_W^2}\right) \mathrm{d}W}_{=1}$$

Trade-off is represented by:

$$V(\mu_W, \sigma_W^2) = \mu - \frac{1}{2}\rho\sigma_W^2$$

Certainty Equivalent in CARA-Gaussian Setup

More generally, multinomial random variables $\mathbf{w} \sim \mathcal{N}(0, \mathbf{\Sigma})$ with a positive definite (co)variance matrix $\mathbf{\Sigma}$. More specifically:

$$E[\exp(\mathbf{w}^{\mathbf{T}}\mathbf{A}\mathbf{w} + \mathbf{b}^{\mathbf{T}}\mathbf{w} + d)] =$$
$$= |\mathbf{I} - 2\mathbf{\Sigma}\mathbf{A}|^{-1/2} \exp\left[\frac{1}{2}\mathbf{b}^{T}(\mathbf{I} - 2\mathbf{\Sigma}\mathbf{A})^{-1}\mathbf{\Sigma}\mathbf{b} + d\right],$$

where:
$\mathbf{A}$ is a symmetric $m \times m$ matrix,
$\mathbf{b}$ is an m-vector, and
d is a scalar.
Note that the left-hand side is only well-defined if $(\mathbf{I} - 2\mathbf{\Sigma}\mathbf{A})$ is positive definite.

Demand for a Risky Asset

asset	payoff	endowment
bond (numeraire)	R	e_0^i
stock	$v \sim \mathcal{N}(E[v\|\cdot], Var[v\|\cdot])$	z^i

- Two assets
- $Px^i + b^i = Pz^i + e_0^i$
- Final wealth:

$$W^i = b^i R + x^i v = (e_0^i + P(z^i - x^i))R + x^i v$$

- Mean: $(e_0^i + P(z^i - x^i))R + xE[v|\cdot]$
- Variance: $(x^i)^2 Var[v|\cdot]$

Demand for a Risky Asset

$$
\begin{aligned}
V(\mu_W, \sigma_W^2) &= \mu_W - \frac{1}{2}\rho^i \sigma_W^2 \\
&= (e_0^i + Pz^i)R + x^i(E[v|\cdot] - PR) - \frac{1}{2}\rho^i Var[v|\cdot](x^i)^2
\end{aligned}
$$

First order condition: $E[v|\cdot] - PR - \rho^i Var[v|\cdot] x^i = 0$

$$
x^i(P) = \frac{E[v|\cdot] - PR}{\rho^i Var[v|\cdot]}
$$

Remarks:

- independent of initial endowment (CARA)
- linearly increasing in investor's expected excess return
- decreasing in investors' variance of the payoff $Var[v|\cdot]$
- decreasing in investors' risk aversion ρ^i
- for $\rho^i \to 0$ investors are risk-neutral, and $x^i \to +\infty$ or $-\infty$

Symmetric Info – Benchmark

Model setup:

- $i \in \{1, ..., I\}$ (types of) traders
- CARA utility function with risk aversion coefficient ρ^i (Let $\eta^i = \frac{1}{\rho^i}$ be trader i's risk tolerance)
- all traders have the same information $v \sim \mathcal{N}(\mu, \sigma_v^2)$
- aggregate demand: $\sum_i^I \frac{E[v]-PR}{\rho^i Var[v]} = \sum_i^I \eta^i \tau_v \{E[v] - PR\}$
  Let $\eta := \frac{1}{I} \sum_i^I \eta^i = \frac{1}{I} \sum_i^I \frac{1}{\rho^i}$ (harmonic mean)
- market clearing: $\eta I \tau_v \{E[v] - PR\} = X^{\text{supply}}$

$$P = \frac{1}{R} \left\{ E[v] - \frac{X^{\text{sup}}}{I \eta \tau_v} \right\}$$

- Expected excess payoff:

$$Q := E[v] - PR = \frac{1}{\eta \tau_v} \frac{X^{\text{sup}}}{I}$$

Symmetric Info – Benchmark

- Trader i's equilibrium demand is:

$$x^i(P) = \frac{\eta^i}{\eta}\frac{X^{\text{sup}}}{I}$$

- Remarks:
  - $\frac{\partial P}{\partial E[v]} = \frac{1}{R} > 0$
  - $\frac{\eta^i}{\eta}$ risk sharing of aggregate endowment:

$$\frac{x^{i*}}{x^{i'*}} = \frac{\eta^i}{\eta^{i'}}$$

  - no endowment effects

REE – Grossman (1976)

Model setup:

- $i \in \{1, ..., I\}$ traders
- CARA utility function with risk aversion coefficient $\rho = \rho^i$ (Let $\eta^i = \frac{1}{\rho^i}$ be trader i's risk tolerance)
- information is dispersed among traders: trader i's signal is $S^i = v + \epsilon^i_S$, where $\epsilon^i_S \sim^{i.i.d.} \mathcal{N}(0, \sigma^2_\epsilon)$

REE – Grossman (1976)

Step 1: Conjecture price function

$$P = \alpha_0 + \alpha_S \bar{S}, \text{ where } \bar{S} = \frac{1}{I}\sum_i^I S^i \text{ (sufficient statistics)}$$

Step 2: Derive posterior distribution

$$\begin{aligned} E[v|S^i, P] = E[v|\bar{S}] &= \lambda E[v] + (1-\lambda)\bar{S} \\ &= \lambda E[v] + (1-\lambda)\frac{P-\alpha_0}{\alpha_S} \end{aligned}$$

$$Var[v|S^i, P] = Var[v|\bar{S}] = \lambda Var[v],$$

$$\text{where } \lambda := \frac{Var[\epsilon]}{I Var[v] + Var[\epsilon]}$$

Step 3: Derive individual demand

$$x^{i*}(P) = \frac{E[v|S^i, P] - P(1+r)}{\rho^i Var[v|S^i, P]}$$

REE – Grossman (1976)

Step 4: Impose market clearing

$$\sum_i^I x^{i*}(P) = X^{\text{supply}}$$

$$P = \underbrace{\frac{\lambda}{1+r}\left(E[v] - \rho^i Var[v]\frac{1}{I}X^{\text{supply}}\right)}_{=\alpha_0} + \underbrace{\frac{1-\lambda}{1+r}}_{=\alpha_S}\bar{S},$$

$$\text{where } \bar{S} = \frac{P-\alpha_0}{\alpha_S}$$

Step 5: Impose rationality
(determine undetermined coefficients α_0, α_S)
N.B.: Price fully reveals sufficient statistic!

Informational (Market) Efficiency

Empirical Literature:

Form	Price reflects
strong	all private and public information
semi strong	all public information
weak	only (past) price information

Theoretical Literature:

Form	Price aggregates/reveals
fully revealing	all private signals
informational efficient	sufficient statistic of signals
partially revealing	a noisy signal of pooled private info
privately revealing	**with** one signal reveals suff. stat.

Informational (Market) Efficiency

- $\overline{\mathbf{S}}$ sufficient statistic for all individual info sets $\{\mathcal{S}^1, ..., \mathcal{S}^I\}$
- Illustration: if one can view price function as

$$P(\cdot) : \{\mathcal{S}^1, ..., \mathcal{S}^I\} \overset{g(\cdot)}{\rightarrow} \overline{\mathbf{S}} \overset{f(\cdot)}{\rightarrow} P$$

- if $f(\overline{\mathbf{S}})$ is invertible, then price is informationally efficient
- if $f(\cdot)$ and $g(\cdot)$ are invertible, then price is fully revealing

Remarks & Paradox

- Grossman (1976) solved it via "full communication equilibria" (Radner 1979's terminology)
- 'unique' info efficient equilibrium (DeMarzo & Skiadas 1998)
- As $I \to \infty$ (risk-bearing capacity), $P \to \frac{1}{R} E[v]$
- Grossman Paradox:
  Individual demand does not depend on individual signal S^is.
  How can all information be reflected in the price?
- Grossman-Stiglitz Paradox:
  Nobody has an incentive to collect information?
- individual demand is independent of wealth (CARA)
- in equilibrium individual demand is independent of price
- equilibrium is not implementable

Noisy REE – Hellwig 1980

Model setup:

- $i \in \{1, ..., I\}$ traders
- CARA utility function with risk aversion coefficient $\rho = \rho^i$ (Let $\eta^i = \frac{1}{\rho^i}$ be trader i's risk tolerance)
- Information is dispersed among traders: trader i's signal is $S^i = v + \epsilon_S^i$, where $\epsilon_S^i \sim^{ind} \mathcal{N}(0, (\sigma_\epsilon^i)^2)$
- noisy asset supply $X^{\text{Supply}} = u$
- Let $\Delta S^i = S^i - E[S^i]$, $\Delta u = u - E[u]$ etc.

Noisy REE – Hellwig (1980)

Step 1: Conjecture price function

$$P = \alpha_0 + \sum_{i}^{I} \alpha_S^i \Delta S^i + \alpha_u \Delta u$$

Step 2: Derive posterior distribution (let's do it only half way through)

$$E[v|S^i, P] = E[v] + \beta_S^i(\alpha)\Delta S^i + \beta_P(\alpha)\Delta P$$

$$Var[v|S^i, P] = \frac{1}{\tau^i_{[v|S^i,P]}} \text{ (independent of signal realization)}$$

Step 3: Derive individual demand

$$x^{i*}(P) = \eta^i \tau^i_{[v|S^i,P]} \{E[v|S^i, P] - P(1+r)\}$$

Noisy REE – Hellwig (1980)

Step 4: Impose market clearing

Total demand = total supply (let $r = 0$)

$$\sum_i^I \eta^i \tau^i_{[v|S^i,P]}(\alpha)\{E[v]+\beta^i_S(\alpha)\Delta S^i-\alpha_0\beta^i_P(\alpha)+[\beta^i_P(\alpha)-1]P\} = u$$

...

$$P\left(S^1,...,S^I,u\right) =$$

$$\frac{\sum_i \left(\eta^i\tau^i_{[v|S^i,P]}\left(\alpha\right)\right)\left[E\left[v\right]-\alpha_0\beta^i_P\left(\alpha\right)+\beta^i_S\left(\alpha\right)\Delta S^i\right]-E\left[u\right]-\Delta u}{\sum_i\left(1-\beta^i_P\left(\alpha\right)\right)\eta^i\tau^i_{[v|S^i,P]}\left(\alpha\right)}$$

Noisy REE – Hellwig (1980)

Step 5: Impose rationality

$$
\begin{aligned}
\alpha_0 &= \frac{\sum_i \left(\eta^i \tau^i_{[v|S^i,P]}(\alpha)\right)\left[E[v] - \alpha_0 \beta^i_P(\alpha)\right] - E[u]}{\sum_i \left(1 - \beta^i_P(\alpha)\right) \eta^i \tau^i_{[v|S^i,P]}(\alpha)} \\
\alpha^i_S &= \frac{\eta^i \tau^i_{[v|S^i,P]}(\alpha)}{\sum_i \left(1 - \beta^i_P(\alpha)\right) \eta^i \tau^i_{[v|S^i,P]}(\alpha)} \beta^i_S(\alpha) \\
\alpha^i_u &= \frac{-1}{\sum_i \left(1 - \beta^i_P(\alpha)\right) \eta^i \tau^i_{[v|S^i,P]}(\alpha)}
\end{aligned}
$$

Solve for root α^* of the problem $\alpha = G(\alpha)$

Noisy REE – Hellwig 1980

Simplify model setup:

- All traders have identical risk aversion coefficient $\rho = 1/\eta$
- Error of all traders' signals ϵ_S^i are i.i.d.

Step 1: Conjecture price function simplifies to:

$$\Delta P = \alpha_S \sum_i^I \frac{1}{I} \Delta S^i + \alpha_u \Delta u$$

Step 2: Derive posterior distribution:

$$E[v|S^i, P] = E[v] + \beta_S(\alpha)\Delta S^i + \beta_P(\alpha)\Delta P$$

$$Var[v|S^i, P] = \frac{1}{\tau} \quad \text{(independent of signal realization)}$$

where β's are projection coefficients.

Noisy REE – Hellwig (1980)

Previous fixed point system simplifies to:

$$\alpha_S = \frac{1}{\sum_i \left(1 - \beta_P(\alpha)\right)} \beta_S(\alpha)$$

$$\alpha_u = \frac{-1}{\eta\tau(\alpha) \sum_i \left(1 - \beta_P(\alpha)\right)}$$

To determine β_S and β_P, invert Co-variance matrix

$$\Sigma\left(S^i, P\right) = \begin{pmatrix} \sigma_v^2 + \sigma_\varepsilon^2 & \alpha_S\left(\sigma_v^2 + \frac{1}{I}\sigma_\varepsilon^2\right) \\ \alpha_S\left(\sigma_v^2 + \frac{1}{I}\sigma_\varepsilon^2\right) & \alpha_S^2\left(\sigma_v^2 + \frac{1}{I}\sigma_\varepsilon^2\right) + \alpha_u^2\sigma_u^2 \end{pmatrix}$$

$$\Sigma^{-1}\left(S^i, P\right) = \frac{1}{D}\begin{pmatrix} \alpha_S^2\left(\sigma_v^2 + \frac{1}{I}\sigma_\varepsilon^2\right) + \alpha_u^2\sigma_u^2 & -\alpha_S\left(\sigma_v^2 + \frac{1}{I}\sigma_\varepsilon^2\right) \\ -\alpha_S\left(\sigma_v^2 + \frac{1}{I}\sigma_\varepsilon^2\right) & \sigma_v^2 + \sigma_\varepsilon^2 \end{pmatrix}$$

$$D = \alpha_S^2 \frac{I-1}{I}\left(\sigma_v^2 + \frac{1}{I}\sigma_\varepsilon^2\right)\sigma_\varepsilon^2 + \alpha_u^2\sigma_u^2\left(\sigma_v^2 + \sigma_\varepsilon^2\right)$$

Noisy REE – Hellwig (1980)

Since $Cov\left[v, P\right] = \alpha_S \sigma_v^2$ and $Cov\left[v, S^i\right] = \sigma_v^2$ leads us to:

$$\beta_P = \frac{1}{D}\alpha_S \frac{I-1}{I}\sigma_v^2\sigma_\epsilon^2$$
$$\beta_S = \frac{1}{D}\alpha_u^2\sigma_u^2\sigma_v^2$$

For conditional variance (precision) from projection theorem:

$$Var\left[v|S^i, P\right] = \frac{1}{D}\left[D\sigma_v^2 - \left(\alpha_u^2\sigma_u^2 + \alpha_S^2\frac{I-1}{I}\sigma_\varepsilon^2\right)\sigma_v^4\right]$$
$$= \frac{1}{D}\left[\alpha_S^2\frac{I-1}{I^2}\sigma_\varepsilon^2 + \alpha_u^2\sigma_u^2\right]\left(\sigma_\varepsilon^2\right)\sigma_v^2$$

Hence:

Noisy REE – Hellwig (1980)

$$\alpha_S = \frac{\alpha_u^2 \sigma_v^2 \sigma_u^2}{(D - \alpha_s \frac{I-1}{I} \sigma_\varepsilon^2 \sigma_v^2) I}$$

$$\alpha_u = -\rho \frac{\left(\alpha_u^2 \sigma_u^2 + \alpha_s^2 \frac{I-1}{I^2} \sigma_\varepsilon^2\right) \sigma_\varepsilon^2 \sigma_v^2}{(D - \alpha_s \frac{I-1}{I} \sigma_\varepsilon^2 \sigma_v^2) I}$$

Trick: solve for $h = -\frac{\alpha_u}{\alpha_S}$ (Recall price signal can be re-written as $\frac{P-\alpha_0}{\alpha_S} = \sum_i^I \frac{1}{I} S^i + \frac{\alpha_u}{\alpha_S} u$) [noise signal ratio]

$$h = \frac{\rho \left(h^2 \sigma_u^2 + \frac{I-1}{I^2} \sigma_\varepsilon^2\right) \sigma_\varepsilon^2 \sigma_v^2}{h^2 \sigma_v^2 \sigma_u^2}$$

$$\underbrace{h}_{\text{increasing in } h} = \underbrace{\rho \sigma_\varepsilon^2 + \frac{\rho}{h^2} \frac{\overbrace{(I-1)\,\sigma_\varepsilon^4}^{>0}}{I^2 \sigma_u^2}}_{\text{decreasing in } h} \quad \Rightarrow \text{ unique } h > \rho \sigma_\varepsilon^2$$

Noisy REE – Hellwig (1980)

Remember that h is increasing in ρ. Back to α_S:

$$\alpha_S = \frac{\alpha_u^2 \sigma_v^2 \sigma_u^2}{\left(D - \alpha_s \frac{I-1}{I} \sigma_\varepsilon^2 \sigma_v^2\right) I} \quad \text{multiply by denominator:}$$

$$\alpha_S D I = \alpha_u^2 \sigma_v^2 \sigma_u^2 + (I-1)\alpha_S^2 \frac{I-1}{I} \sigma_\varepsilon^2 \sigma_v^2$$

$$\Leftrightarrow \alpha_S = \frac{1}{D}\left[\frac{1}{I}\alpha_u^2 \sigma_v^2 \sigma_u^2 + \alpha_S^2 \frac{I-1}{I} \sigma_\varepsilon^2 \sigma_v^2\right]$$

Sub in $D = \dots$

$$\alpha_S = \frac{\frac{\alpha_u^2}{\alpha_s^2} \sigma_v^2 \sigma_u^2 + (I-1)\sigma_\varepsilon^2 \sigma_v^2}{(I-1)\left(\sigma_v^2 + \frac{1}{I}\sigma_\varepsilon^2\right)\sigma_\varepsilon^2 + I \frac{\alpha_u^2}{\alpha_s^2} \sigma_u^2 \left(\sigma_v^2 + \sigma_\varepsilon^2\right)} \Rightarrow \text{ unique } \alpha_S$$

This proves existence and uniqueness of the NREE!

Characterization of NREE

Recall that:

$$Var\left[v|S^i,P\right]=\frac{1}{D}\left[\alpha_S^2\frac{I-1}{I^2}\sigma_\varepsilon^2+\alpha_u^2\sigma_u^2\right]\sigma_\varepsilon^2\sigma_v^2,\ \text{and}$$

$$\alpha_S=\frac{1}{D}\left[\frac{1}{I}\alpha_u^2\sigma_v^2\sigma_u^2+\alpha_S^2\frac{I-1}{I}\sigma_\varepsilon^2\sigma_v^2\right]$$

Hence,

$$\alpha_S=Var\left[v|S^i,P\right]\frac{\left[\frac{1}{I}\alpha_u^2\sigma_u^2+\alpha_s^2\frac{I-1}{I}\sigma_\varepsilon^2\right]}{\left[\alpha_S^2\frac{I-1}{I^2}\sigma_\varepsilon^2+\alpha_u^2\sigma_u^2\right]\sigma_\varepsilon^2}\ \text{(notice } I^2 \text{ square)}$$

$$\alpha_S=Var\left[v|\cdot\right]\frac{1}{\sigma_\varepsilon^2}\frac{\left[\frac{I}{I-1}h^2\sigma_u^2+I\sigma_\varepsilon^2\right]}{\left[\sigma_\varepsilon^2+\frac{I^2}{I-1}h^2\sigma_u^2\right]}$$

$$=Var\left[v|\cdot\right]\frac{1}{\sigma_\varepsilon^2}\frac{\left[\frac{I}{I-1}h^2\sigma_u^2+\frac{1}{I}\sigma_\varepsilon^2+\left(I-\frac{1}{I}\right)\sigma_\varepsilon^2\right]}{\left[\sigma_\varepsilon^2+\frac{I^2}{I-1}h^2\sigma_u^2\right]}$$

Characterization of NREE

$$\alpha_S = Var\left[v|S^i, P\right]\frac{1}{\sigma_\varepsilon^2}\left[\frac{1}{I} + \frac{\left(I - \frac{1}{I}\right)\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \frac{I^2}{I-1}h^2\sigma_u^2}\right]$$

$$= Var\left[v|S^i, P\right]\frac{\tau_\varepsilon}{I}\left[1 + (I+1)\underbrace{\left((I-1)\frac{\tau_u}{\tau_u + \frac{I^2}{I-1}h^2\tau_\varepsilon}\right)}_{:=\theta}\right]$$

$$= Var\left[v|S^i, P\right]\frac{\tau_\varepsilon}{I}\left[1 + (I+1)\theta\right],$$

where θ is decreasing in ρ (h is increasing)

Characterization of NREE

$$Var\left[v|S^i,P\right] = \frac{1}{D}\left[\alpha_S^2\frac{I-1}{I^2}\sigma_\varepsilon^2 + \alpha_u^2\sigma_u^2\right]\sigma_\varepsilon^2\sigma_v^2$$

$$= \frac{\left[\alpha_S^2\frac{I-1}{I^2}\sigma_\varepsilon^2 + \alpha_u^2\sigma_u^2\right]\sigma_\varepsilon^2\sigma_v^2}{\alpha_S^2\frac{I-1}{I}\left(\sigma_v^2 + \frac{1}{I}\sigma_\varepsilon^2\right)\sigma_\varepsilon^2 + \alpha_u^2\sigma_u^2\left(\sigma_v^2+\sigma_\varepsilon^2\right)}$$

$$= \frac{\left[\frac{I-1}{I^2}\sigma_\varepsilon^2 + h^2\sigma_u^2\right]\sigma_\varepsilon^2\sigma_v^2}{\frac{I-1}{I}\left(\sigma_v^2 + \frac{1}{I}\sigma_\varepsilon^2\right)\sigma_\varepsilon^2 + h^2\sigma_u^2\left(\sigma_v^2+\sigma_\varepsilon^2\right)} = \ldots$$

$$\frac{1}{Var\left[v|S^i,P\right]} = \tau_v + \tau_\varepsilon + \theta\tau_\varepsilon$$

Interpretation

$\theta = (I-1)\,\tau_u/(\tau_u + \frac{I^2}{I-1}h^2\tau_\varepsilon)$ measure of info efficiency

$\sigma_u^2 \to \infty$ ($\tau_u \to 0$): $\theta \to 0$ price is uninformative (Walras. equ.)

$\sigma_u^2 \to 0$ ($\tau_u \to \infty$): $\theta \to 1$ price is informationally efficient

Remarks to Hellwig (1980)

- Since $\alpha_u^2 \neq 0$, $\beta_S \neq 0$, i.e. agents condition on their signal
- As risk aversion of trader increases the informativeness of price θ declines
- Price informativeness increases in precision of signal τ_ε and declines in the amount of noise trading σ_u^2
- Negative supply shock leads to a larger price increase compared to a Walrasian equilibrium, since traders wrongly partially attribute it to a good realization of v
- Diamond and Verrecchia (1981) is similar except that endowment shocks of traders serve as asymmetric information

Endogenous Info Acquisition Grossman-Stiglitz (1980)

Model setup:

- $i \in \{1, ..., I\}$ traders
- CARA utility function with risk aversion coefficient ρ (Let $\eta = \frac{1}{\rho}$ be traders' risk tolerance)
- no information aggregation – two groups of traders
  - Informed traders who have the same signal S: $S = v + \epsilon_S$ with $\epsilon_S \sim \mathcal{N}(0, \sigma_\epsilon^2)$
  - Uninformed traders have no signal
- FOCUS on information acquisition

Noisy REE – Grossman-Stiglitz

Step 1: Conjecture price function

$$P = \alpha_0 + \alpha_S \Delta S + \alpha_u \Delta u$$

Step 2: Derive posterior distribution

- for informed traders:

$$E[v|S,P] = E[v|S] = E[v] + \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon} \Delta S$$

$$\tau_{[v|S]} = \tau_v + \tau_\varepsilon$$

- for uninformed traders:

$$E[v|P] = E[v] + \frac{\alpha_S \sigma_v^2}{\alpha_S^2(\sigma_v^2 + \sigma_\varepsilon^2) + \alpha_u^2 \sigma_u^2} \Delta P$$

$$Var[v|P] = \sigma_v^2 \left(1 - \frac{\alpha_S^2 \sigma_v^2}{\alpha_S^2(\sigma_v^2 + \sigma_\epsilon^2) + \alpha_u^2 \sigma_u^2}\right)$$

or: $\tau_{[v|P]} = \tau_v + \underbrace{\frac{\tau_u}{\tau_u + h^2 \tau_\varepsilon}}_{:=\phi \in [0,1]} \tau_\varepsilon$, where $h = -\frac{\alpha_u}{\alpha_S}$

Noisy REE – Grossman-Stiglitz

After some algebra we get:

$$E[v|P] = E[v] + \frac{1}{\alpha_S}\frac{\phi\tau_\varepsilon}{\tau_v + \phi\tau_\varepsilon}\Delta P$$

Step 3: Derive individual demand

$$x^I(P,S) = \eta^I[\tau_v + \tau_\varepsilon]\left[E[v] + \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon}\Delta S - P\right]$$

$$x^U(P) = \eta^U[\tau_v + \phi\tau_\varepsilon]\left[E[v] + \frac{1}{\alpha_S}\frac{\phi\tau_\varepsilon}{\tau_v + \phi\tau_\varepsilon}\Delta P - P\right]$$

Step 4: Impose market clearing

$$\underbrace{\lambda^I\eta^I[\tau_v + \tau_\varepsilon]}_{:=\nu^I}\left[E[v] + \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon}\Delta S - P\right] +$$

$$\underbrace{\left(1 - \lambda^I\right)\eta^U[\tau_v + \phi\tau_\varepsilon]}_{:=\nu^U}\left[E[v] + \frac{1}{\alpha_S}\frac{\phi\tau_\varepsilon}{\tau_v + \phi\tau_\varepsilon}\Delta P - P\right] = u$$

Noisy REE – Grossman-Stiglitz

$$P(S,u) = \frac{\left(\nu^I + \nu^U\right) E[v] + \nu^I \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon} \Delta S - \frac{1}{\alpha_S} \frac{\Phi \tau_\varepsilon}{\tau_v + \phi \tau_\varepsilon} \alpha_0 \nu^U - E[u] - \Delta u}{\nu^U \left(1 - \frac{1}{\alpha_S} \frac{\Phi \tau_\varepsilon}{\tau_v + \phi \tau_\varepsilon}\right) + \nu^I}$$

Hence,

$$h = -\frac{\alpha_u}{\alpha_S} = \left[\nu^I \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon}\right]^{-1} = \frac{1}{\lambda^I \eta^I \tau_\varepsilon}$$

$$\phi = \frac{\tau_u \tau_\varepsilon}{\tau_u \tau_\varepsilon + \frac{1}{\left(\lambda^I \eta^I\right)^2}}$$

Remarks:

- As $Var[u] \searrow 0$, $\phi \nearrow 1$
- If signal is more precise (τ_ε is increasing) then ϕ increases (since informed traders are more aggressive)
- Increases in λ^I and η^I also increase ϕ

Noisy REE – Grossman-Stiglitz

Step 5: Impose rationality

Solve for coefficients:

$$\alpha_0 = E[v] - \frac{1}{\nu^I + \nu^U} E[u]$$

$$\alpha_S = \frac{1}{\nu^U \left(1 - \frac{1}{\alpha_S} \frac{\Phi \tau_\varepsilon}{\tau_v + \Phi \tau_\varepsilon}\right) + \nu^I} \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon} \nu^I = \frac{\lambda^I \eta^I + \lambda^U \eta^U \Phi}{\nu^I + \nu^U} \tau_\varepsilon$$

$$\alpha_u = -\frac{1}{\nu^I + \nu^U} \left(1 + \frac{\lambda^U \eta^U}{\lambda^I \eta^I} \Phi\right)$$

Finally let's calculate:

$$\frac{\tau_{[v|S]}}{\tau_{[v|P]}} = \frac{\tau_v + \tau_\varepsilon}{\tau_v + \phi \tau_\varepsilon} = 1 + \frac{(1 - \phi) \tau_\varepsilon}{\tau_v + \phi \tau_\varepsilon}$$

Information Acquisition Stage – Grossman-Stiglitz (1980)

- Aim: endogenize λ^I
- Recall:

$$x^i = \eta^i \tau_{[Q|S]} E[Q|S], \text{ where } Q = v - RP \text{ is excess payoff}$$

- Final wealth:

$$W^i = \eta^i Q \tau_{[Q|S]} E[Q|S] + (Pu^i + e_0^i) R$$

(CARA $\Rightarrow$ we can ignore second term)
Note W^i is product of two normally distributed variables.
Use formula of Slide 7 **or** follow following steps:

- Conditional on S, wealth is normally distributed:

$$E[W|S] = \eta \tau_{[Q|S]} E[Q|S]^2$$
$$Var[W|S] = \eta^2 \tau_{[Q|S]} E[Q|S]^2$$

Information Acquisition Stage – Grossman-Stiglitz (1980)

Expected utility conditional on S:

$$E[U(W)|S] = -\exp\left\{-\frac{1}{\eta}[\eta\tau_{[Q|S]}E[Q|S]^2 - \frac{1}{2}\eta\tau_{[Q|S]}E[Q|S]^2]\right\}$$
$$= -\exp\left\{-\frac{1}{2}\tau_{[Q|S]}E[Q|S]^2\right\}$$

Integrate over possible S to get the ex-ante utility:

w.l.o.g. we can assume that $S = Q + \epsilon$

Normal density $f(S) = \sqrt{\frac{\tau_S}{2\pi}}\exp\{-\frac{1}{2}\tau_S[\Delta S]^2\}$

$$E\left[U\left(W\right)\right] =$$
$$-\int_S \sqrt{\frac{\tau_S}{2\pi}}\exp\left\{-\frac{1}{2}\left[\tau_{[Q|s]}E\left[Q|S\right]^2 + \tau_S\left(\Delta S\right)^2\right]\right\}\mathrm{d}S$$

Information Acquisition Stage – Grossman-Stiglitz (1980)

Term in square bracket is:

$$\left[(\tau_Q + \tau_\varepsilon)\left(E[Q] + \frac{\tau_\varepsilon}{\tau_Q + \tau_\varepsilon}\Delta S\right)^2 + \frac{\tau_Q \tau_\varepsilon}{\tau_Q + \tau_\varepsilon}(\Delta S)^2\right]$$

which simplifies to:

$$\tau_Q E[Q]^2 + \tau_\varepsilon (\Delta S + E[Q])^2$$

Hence:

$$E[U(W)] = \\ -\exp\left\{-\frac{\tau_Q E[Q]^2}{2}\right\}\int_S \sqrt{\frac{\tau_S}{2\pi}} e^{-\frac{1}{2}\left[\tau_\varepsilon (\Delta S + E[Q])^2\right]} \mathrm{d}S$$

Define:

$$y := \sqrt{\tau_\varepsilon}(\Delta S + E[Q])$$

Information Acquisition Stage – Grossman-Stiglitz (1980)

$$E\left[U\left(W\right)\right]$$
$$= -\exp\left\{-\frac{\tau_Q E\left[Q\right]^2}{2}\right\}\sqrt{\frac{\tau_S}{\tau_\varepsilon}}\underbrace{\int_S -\sqrt{\frac{\tau_\epsilon}{2\pi}}\exp\left\{-\frac{1}{2}y^2\right\}\mathrm{d}S}_{=1}$$

$$\text{Let: } k = -\exp\left\{-\frac{\tau_Q E\left[Q\right]^2}{2}\right\}\sqrt{\tau_Q}$$

$$\text{Note: } \tau_S = \frac{\tau_Q \tau_\varepsilon}{\tau_Q + \tau_\varepsilon}$$

Hence:

$$E\left[U\left(W\right)\right] = \frac{k}{\sqrt{\tau_{[Q|S]}}} = \frac{k}{\sqrt{\tau_Q + \tau_\varepsilon}}$$

Willingness to Pay for Signal
General Problem (**No** Price Signal)

- Without price signal p and signal S, expected utility:

$$E\left[U\left(W\right)\right] = \frac{k}{\sqrt{\tau_Q}}$$

- If the agent buys a signal at a price of m_S his expected utility is:

$$\begin{aligned} E\left[U\left(W - m_S\right)\right] &= E\left[-\exp\left(-\rho\left(W - m_S\right)\right)\right] \\ &= E\left[-\exp\left(-\rho\left(W\right)\right)\exp\left(\rho m_S\right)\right] \\ &= \frac{k}{\sqrt{\tau_{[Q|S]}}}\exp\left(\rho m_S\right) \end{aligned}$$

- Agent is indifferent when:

$$\frac{k}{\sqrt{\tau_Q}} = \frac{k}{\sqrt{\tau_{[Q|S]}}}\exp\left(\rho m_S\right)$$

Willingness to Pay for Signal General Problem (**No** Price Signal)

- Hence willingness to pay is:

$$m_S = \eta \ln \left(\sqrt{\frac{\tau_{[Q|S]}}{\tau_Q}} \right)$$

Willingness to pay depends on the improvement in precision

Information Acquisition Stage – Grossman-Stiglitz (1980)

- Every agent has to be indifferent between being informed or not. The cost of the signal is:

$$c = \eta \ln \left(\sqrt{\frac{\tau[v|S]}{\tau[v|P]}} \right) = \eta \ln \left(\sqrt{\frac{\tau_v + \tau_\varepsilon}{\tau_v + \phi \tau_\varepsilon}} \right)$$

(previous slide)

- This determines ϕ:

$$\phi = \frac{\tau_u \tau_\varepsilon}{\tau_u \tau_\varepsilon + \left(\frac{1}{\lambda^I \eta^I} \right)^2}, \text{ which pins down} \lambda^I$$

- Comparative Statics (using IFT):
  - $c \nearrow \Rightarrow \phi \searrow$
  - $\eta \nearrow \Rightarrow \phi \nearrow$ (extreme case: risk-neutrality)
  - $\tau_\varepsilon \nearrow \Rightarrow \phi \nearrow$
  - $\sigma_u^2 \nearrow \Rightarrow \phi \rightarrow$ (number of informed traders $\nearrow$)
  - $\sigma_u^2 \searrow 0 \Rightarrow$ no investor purchases a signal

Information Acquisition Stage

- Further extensions:
  - Purchase signals with different precisions (Verrecchia 1982)
  - Optimal sale of information
    - Photocopied (newsletter) or individualistic signal (Admati & Pfleiderer)
    - Indirect versus direct (Admati & Pfleiderer)

Endogenizing Noise Trader Demand

- Endowment shocks or outside opportunity shocks that are correlated with asset
- Welfare analysis
  - more private information $\rightarrow$ adverse selection
  - more public information $\rightarrow$ Hirshleifer effect (e.g. genetic testing)
- See papers by Spiegel, Bhattacharya & Rohit, and Vives (2006)

Tips & Tricks

- risk-neutral competitive fringe observing limit order book L

$$p = E[v|L(\cdot)]$$

- separates risk-sharing from informational aspects