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LECTURE 06:
MEAN-VARIANCE ANALYSIS & CAPM

Overview

1. Introduction:
   Simple CAPM with quadratic utility functions
   (from beta-state price equation)
2. Traditional Derivation of CAPM
   - Demand: Portfolio Theory
   - Aggregation: Fund Separation Theorem
   - Equilibrium: CAPM

   for given prices/returns
3. Modern Derivation of CAPM
   - Projections
   - Pricing Kernel and Expectation Kernel
4. Testing CAPM
5. Practical Issues – Black-Litterman

Recall State-price Beta model

Recall:

$$E[R^h] - R^f = \beta^h E\left[R^* - R^f\right]$$

Where $\beta^h := \frac{\text{cov}[R^*,R^h]}{\text{var}[R^*]}$

very general – but what is R^* in reality?

Simple CAPM with Quadratic Expected Utility

1. All agents are identical
   - Expected utility $U(x_0, x_1) = \sum_s \pi_s u(x_0, x_s) \Rightarrow m = \frac{\partial_1 u}{E[\partial_0 u]}$
   - Quadratic $u(x_0, x_1) = v_0(x_0) - (x_1 - \alpha)^2$
     - $\Rightarrow \partial_1 u = \left[-2\left(x_{1,1} - \alpha\right), \dots, -2\left(x_{S,1} - \alpha\right)\right]$
   - Excess return

$$E[R^h] - R^f = -\frac{\mathrm{cov}[m, R^h]}{E[m]} = -\frac{R^f \mathrm{cov}[\partial_1 u, R^h]}{E[\partial_0 u]}$$

$$= -\frac{R^f \mathrm{cov}[-2(x_1 - \alpha), R^h]}{E[\partial_0 u]} = R^f \frac{2\mathrm{cov}[x_1, R^h]}{E[\partial_0 u]}$$

   - Also holds for market portfolio

$$\frac{E[R^h] - R^f}{E[R^{mkt}] - R^f} = \frac{\mathrm{cov}[x_1, R^h]}{\mathrm{cov}[x_1, R^{mkt}]}$$

Simple CAPM with Quadratic Expected Utility

$$\frac{E[R^h]-R^f}{E[R^{mkt}]-R^f}=\frac{\mathrm{cov}[x_1,R^h]}{\mathrm{cov}[x_1,R^{mkt}]}$$

2. Homogenous agents + Exchange economy
   $\Rightarrow x_1 =$ aggr. endowment and is perfectly correlated with R^m

$$\frac{E[R^h]-R^f}{E[R^{mkt}]-R^f}=\frac{\mathrm{cov}[R^{mkt},R^h]}{\mathrm{var}[R^{mkt}]}$$

Since $\beta^h=\frac{\mathrm{cov}[R^h,R^{mkt}]}{\mathrm{var}[R^{mkt}]}$

Market Security Line

$$E[R^h]=R^f+\beta^h\{E[R^{mkt}]-R^f\}$$

NB: $R^*=R^f\frac{a+b_1R^{mkt}}{a+b_1R^f}$ in this case $(b_1<0)$!

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Definition: Mean-Variance Dominance & Efficient Frontier

- Asset (portfolio) A *mean-variance dominates* asset (portfolio) B if $\mu_A \geq \mu_B$ and $\sigma_A < \sigma_B$ or if $\mu_A > \mu_B$ while $\sigma_A \leq \sigma_B$.
- *Efficient frontier*: loci of all non-dominated portfolios in the mean-standard deviation space.
  By definition, no ("rational") mean-variance investor would choose to hold a portfolio not located on the efficient frontier.

Expected Portfolio Returns & Variance

- Expected returns (linear)
  - $\mu^h := E[r^h] = \boldsymbol{w}^{h\prime}\boldsymbol{\mu}$, where each $w^j = \frac{h^j}{\sum_j h^j}$

Everything is in returns (like all prices =1)

- Variance
  - $\sigma_h^2 := \mathrm{var}[r_h] = \boldsymbol{w}'V\boldsymbol{w}$

$$= \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \geq 0$$

Illustration of 2 Asset Case

- For certain weights: w_1 and $1 - w_1$

$$\mu_h = w_1\mu_1 + (1 - w_1)\mu_2$$

$$\sigma_h^2 = w_1^2\sigma_1^2 + (1 - w_1)^2\sigma_2^2 + 2w_1(1 - w_1)\rho_{12}\sigma_1\sigma_2$$

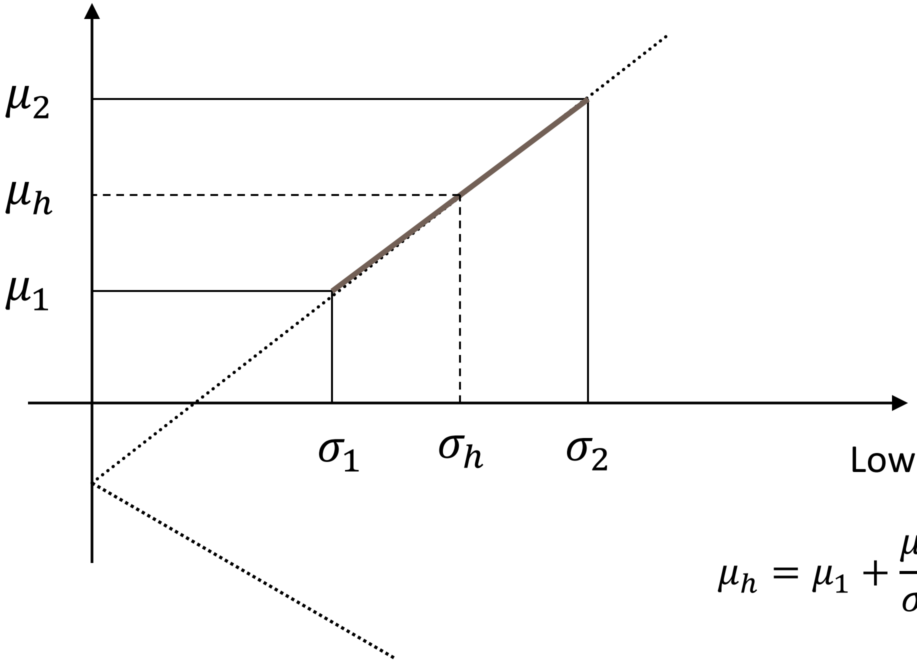
(Specify σ_h^2 and one gets weights and μ_h's)

- Special cases [w_1 to obtain certain σ_h]
  - $\rho_{12} = 1 \Rightarrow w_1 = \frac{\pm\sigma_h - \sigma_2}{\sigma_1 - \sigma_2}$
  - $\rho_{12} = -1 \Rightarrow w_1 = \frac{\pm\sigma_h + \sigma_2}{\sigma_1 + \sigma_2}$

For $\rho_{12} = 1 \Rightarrow w_1 = \frac{\pm\sigma_h - \sigma_2}{\sigma_1 - \sigma_2}$

$$\sigma_h = |w_1\sigma_1 + (1 - w_1)\sigma_2|$$

$$\mu_h = w_1\mu_1 + (1 - w_1)\mu_2 = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}(\pm\sigma_h - \sigma_1)$$



Lower part is irrelevant

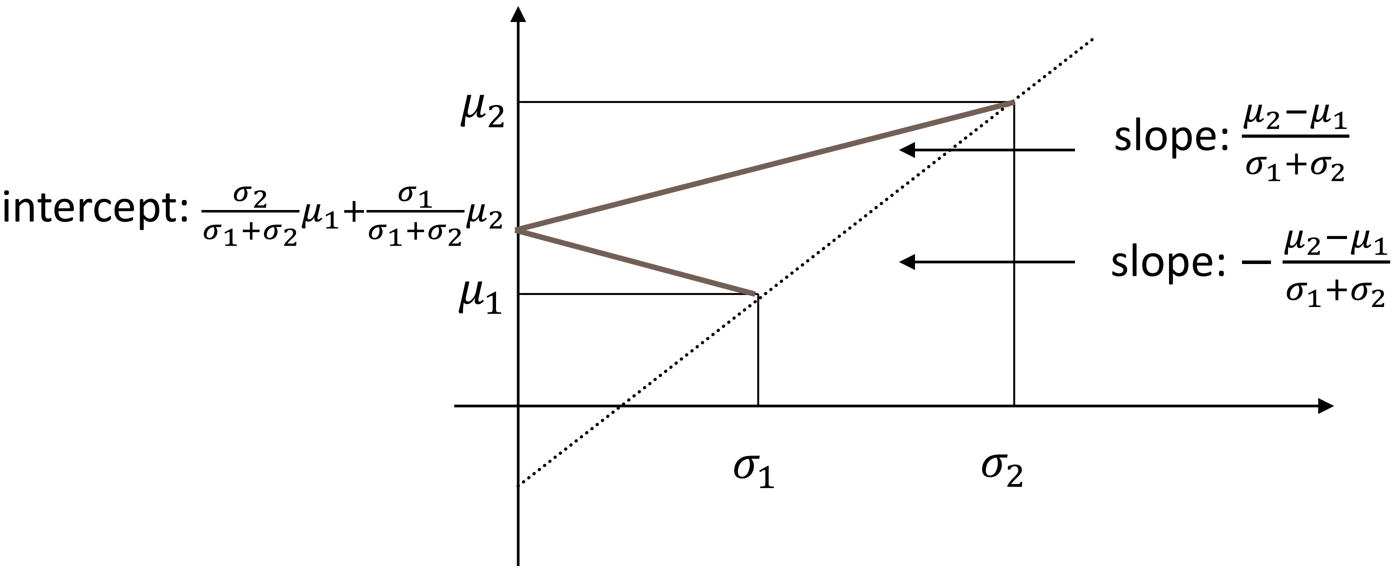
$$\mu_h = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}(-\sigma_h - \sigma_1)$$

The Efficient Frontier: Two Perfectly Correlated Risky Assets

For $\rho_{12} = -1 \Rightarrow w_1 = \frac{\pm\sigma_p + -\sigma_2}{\sigma_1+\sigma_2}$

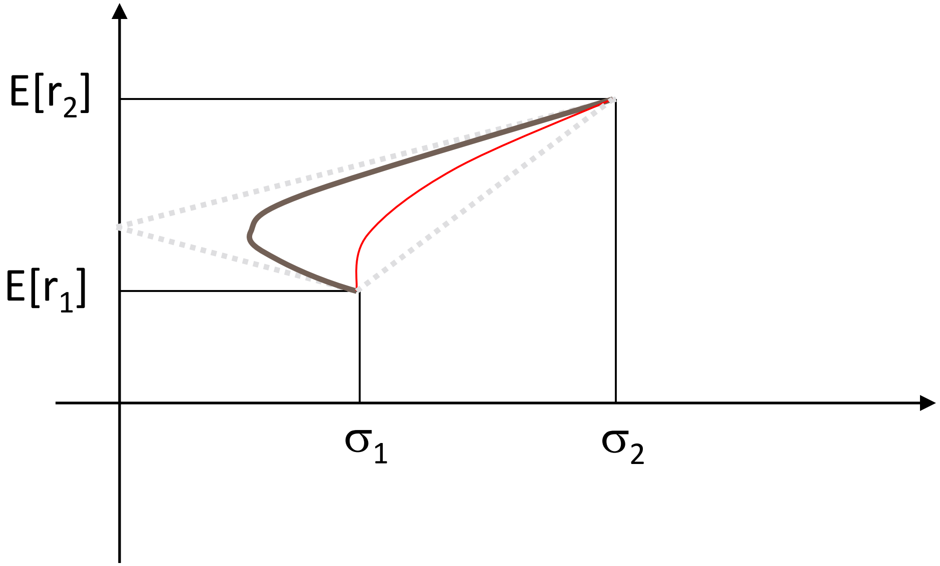
$$\sigma_h = |w_1\sigma_1 - (1 - w_1)\sigma_2|$$

$$\mu_h = w_1\mu_1 + (1 - w_1)\mu_2 = \frac{\sigma_2}{\sigma_1 + \sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1 + \sigma_2}\mu_2 \pm \frac{\mu_2 - \mu_1}{\sigma_1 + \sigma_2}\sigma_p$$



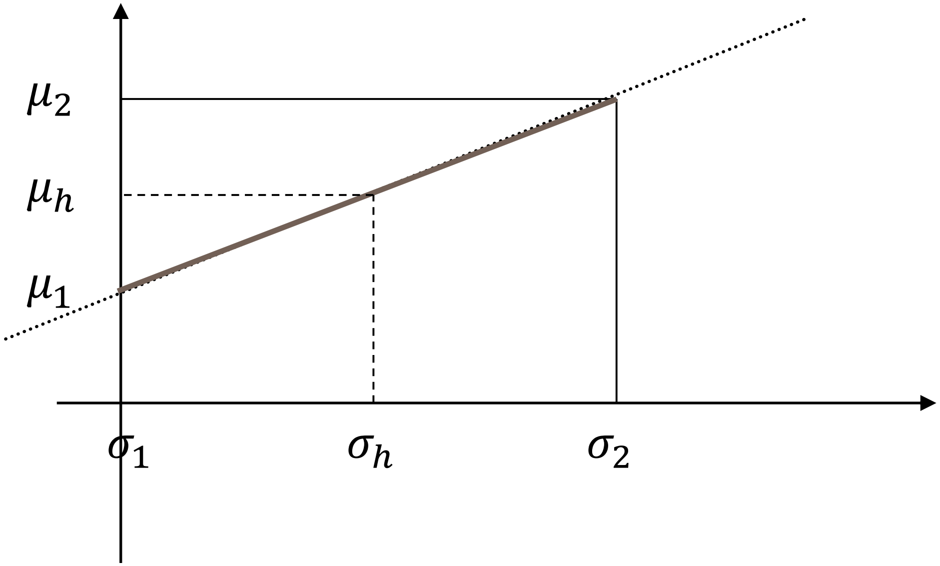
The Efficient Frontier: Two Perfectly Negative Correlated Risky Assets

For $\rho_{12} \in (-1,1)$



The Efficient Frontier: Two Imperfectly Correlated Risky Assets

For $\sigma_1 = 0$

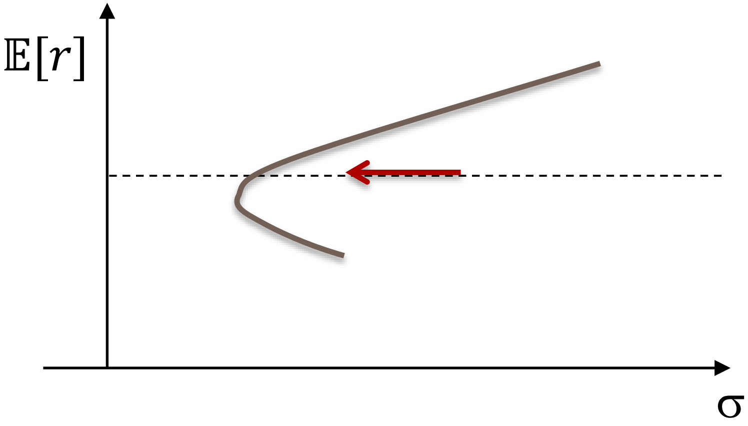


The Efficient Frontier: One Risky and One Risk-Free Asset

Efficient frontier with n risky assets

- *A frontier portfolio is one which displays minimum variance among all feasible portfolios with the same expected portfolio return.*

- $\min_{\boldsymbol{w}} \frac{1}{2} \boldsymbol{w}' V \boldsymbol{w}$
  - $\lambda: \boldsymbol{w}'\boldsymbol{\mu} = \mu^h, \qquad \left(\sum_j w_j\, \mathbb{E}[\tilde{r}_i] = \mu^h\right)$
  - $\gamma: \boldsymbol{w}'\boldsymbol{1} = 1, \qquad \left(\sum_j w_j = 1\right)$



- Result: Portfolio weights are linear in expected portfolio return $w_h = \boldsymbol{g} + \boldsymbol{h}\mu^h$
  - If $\mu^h = 0, w_h = \boldsymbol{g}$
  - If $\mu^h = 1, w_h = \boldsymbol{g} + \boldsymbol{h}$
    - Hence, $\boldsymbol{g}$ and $\boldsymbol{g} + \boldsymbol{h}$ are portfolios on the frontier

$$\frac{\partial \mathcal{L}}{\partial w} = V\boldsymbol{w} - \lambda\boldsymbol{\mu} - \gamma\mathbf{1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \mu^h - \boldsymbol{w}'\boldsymbol{\mu} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \gamma} = 1 - \boldsymbol{w}'\mathbf{1} = 0$$

The first FOC can be written as:

$$V\boldsymbol{w} = \lambda\boldsymbol{\mu} + \gamma\mathbf{1}$$
$$\boldsymbol{w} = \lambda V^{-1}\boldsymbol{\mu} + \gamma V^{-1}\mathbf{1}$$
$$\boldsymbol{\mu}'\boldsymbol{w} = \lambda(\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu}) + \gamma(\boldsymbol{\mu}'V^{-1}\mathbf{1})$$

skip

- Noting that $\boldsymbol{\mu}'\boldsymbol{w}_h = w_h'$, combining 1st and 2nd FOC

$$\mu_h = \boldsymbol{\mu}'\boldsymbol{w}_h = \lambda\,\underbrace{(\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu})}_{B} + \gamma\,\underbrace{(\boldsymbol{\mu}'V^{-1}\mathbf{1})}_{A}$$

- Pre-multiplying the 1st FOC by 1 yields

$$\mathbf{1}'\boldsymbol{w}_h = \boldsymbol{w}_h'\mathbf{1} = \lambda(\mathbf{1}'V^{-1}\boldsymbol{\mu} + \gamma(\mathbf{1}'V^{-1}\mathbf{1}) = 1$$

$$1 = \lambda\underbrace{(\mathbf{1}'V^{-1}\boldsymbol{\mu})}_{A} + \gamma\,\underbrace{(\mathbf{1}'V^{-1}\mathbf{1})}_{C}$$

- Solving for λ, γ

$$\lambda = \frac{C\mu^h - A}{D}, \qquad \gamma = \frac{B - A\mu^h}{D}$$

$$D = BC - A^2$$

skip

- Hence, $\boldsymbol{w}_h = \lambda V^{-1}\boldsymbol{\mu} + \gamma V^{-1}\mathbf{1}$ becomes

$$\boldsymbol{w}_h = \frac{C\mu^h - A}{D} V^{-1}\boldsymbol{\mu} + \frac{B - A\mu^h}{D} V^{-1}\mathbf{1}$$

$$= \underbrace{\frac{1}{D}[B(V^{-1}\mathbf{1}) - A(V^{-1}\boldsymbol{\mu})]}_{\boldsymbol{g}} + \underbrace{\frac{1}{D}[C(V^{-1}\boldsymbol{\mu}) - A(V^{-1}\mathbf{1})]}_{\boldsymbol{h}}\mu^h$$

- Result: Portfolio weights are linear in expected portfolio return $\boldsymbol{w}_h = \boldsymbol{g} + \boldsymbol{h}\mu^h$
  - If $\mu^h = 0, \boldsymbol{w}_h = \boldsymbol{g}$
  - If $\mu^h = 1$, $\boldsymbol{w}_h = \boldsymbol{g} + \boldsymbol{h}$
    - Hence, $\boldsymbol{g}$ and $\boldsymbol{g} + \boldsymbol{h}$ are portfolios on the frontier

skip

Characterization of Frontier Portfolios

- Proposition: *The entire set of frontier portfolios can be generated by ("are convex combinations" $\boldsymbol{g}$ of) and $\boldsymbol{g}+\boldsymbol{h}$.*

- Proposition: *The portfolio frontier can be described as convex combinations of any two frontier portfolios, not just the frontier portfolios $\boldsymbol{g}$ and $\boldsymbol{g}+\boldsymbol{h}$.*

- Proposition: *Any convex combination of frontier portfolios is also a frontier portfolio.*

skip

…Characterization of Frontier Portfolios…

- For any portfolio on the frontier,
  $\sigma^2(\mu^h) = [\boldsymbol{g} + \boldsymbol{h}\mu^h]'V[\boldsymbol{g} + \boldsymbol{h}\mu^h]$
  with $\boldsymbol{g}$ and $\boldsymbol{h}$ as defined earlier.

  Multiplying all this out and some algebra yields:

$$\sigma^2(\mu^h) = \frac{C}{D}\left[\mu^h - \frac{A}{C}\right]^2 + \frac{1}{C}$$

skip

…Characterization of Frontier Portfolios…

i. the expected return of the minimum variance portfolio is $\frac{A}{C}$;

ii. the variance of the minimum variance portfolio is given by $\frac{1}{C}$;

iii. Equation $\sigma^2(\mu^h) = \frac{C}{D}\left[\mu^h - \frac{A}{C}\right]^2 + \frac{1}{C}$ is a
   - parabola with vertex $\left(\frac{1}{C}, \frac{A}{C}\right)$ in the expected return/variance space
   - hyperbola in the expected return/standard deviation space.

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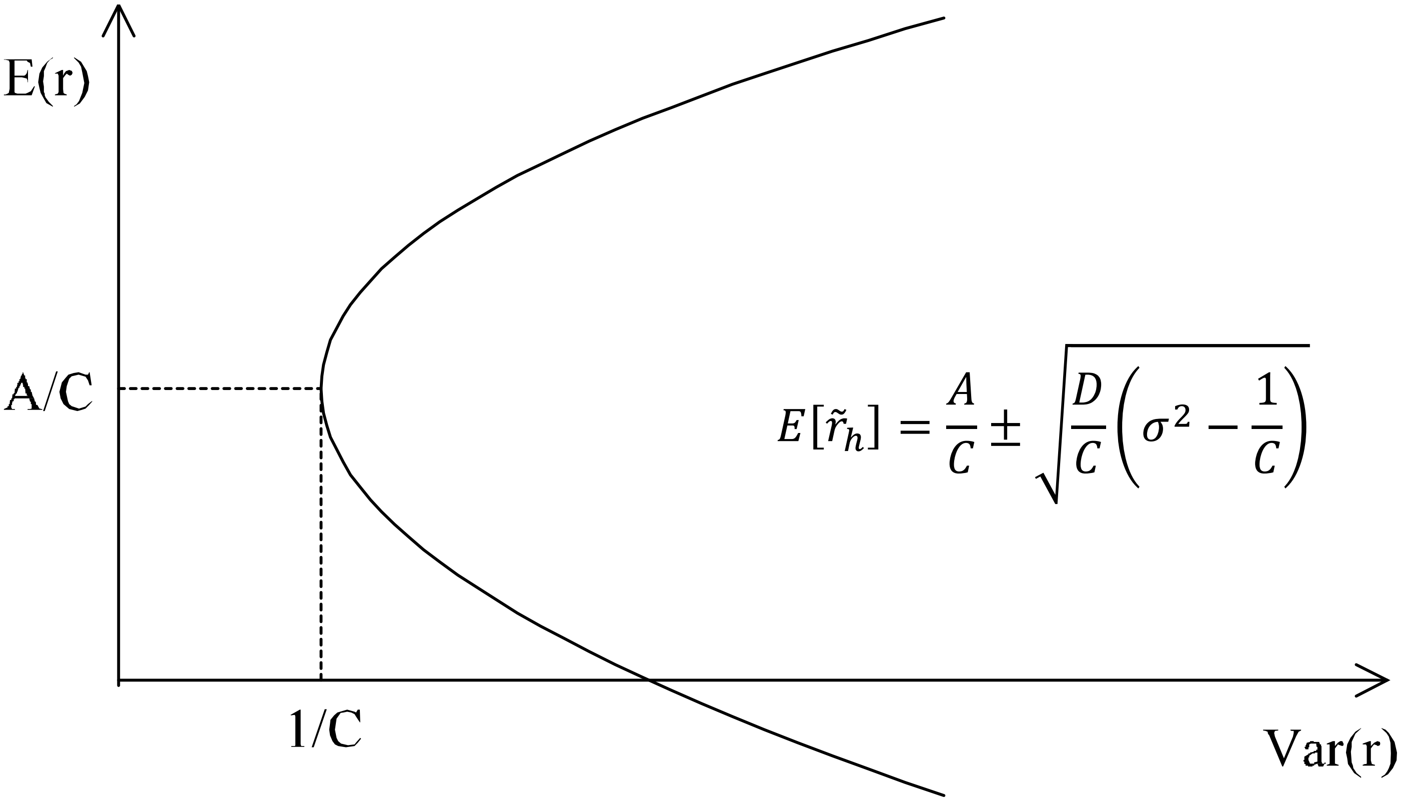


$$E[\tilde{r}_h] = \frac{A}{C} \pm \sqrt{\frac{D}{C}\left(\sigma^2 - \frac{1}{C}\right)}$$

Figure 6-3 The Set of Frontier Portfolios: Mean/Variance Space

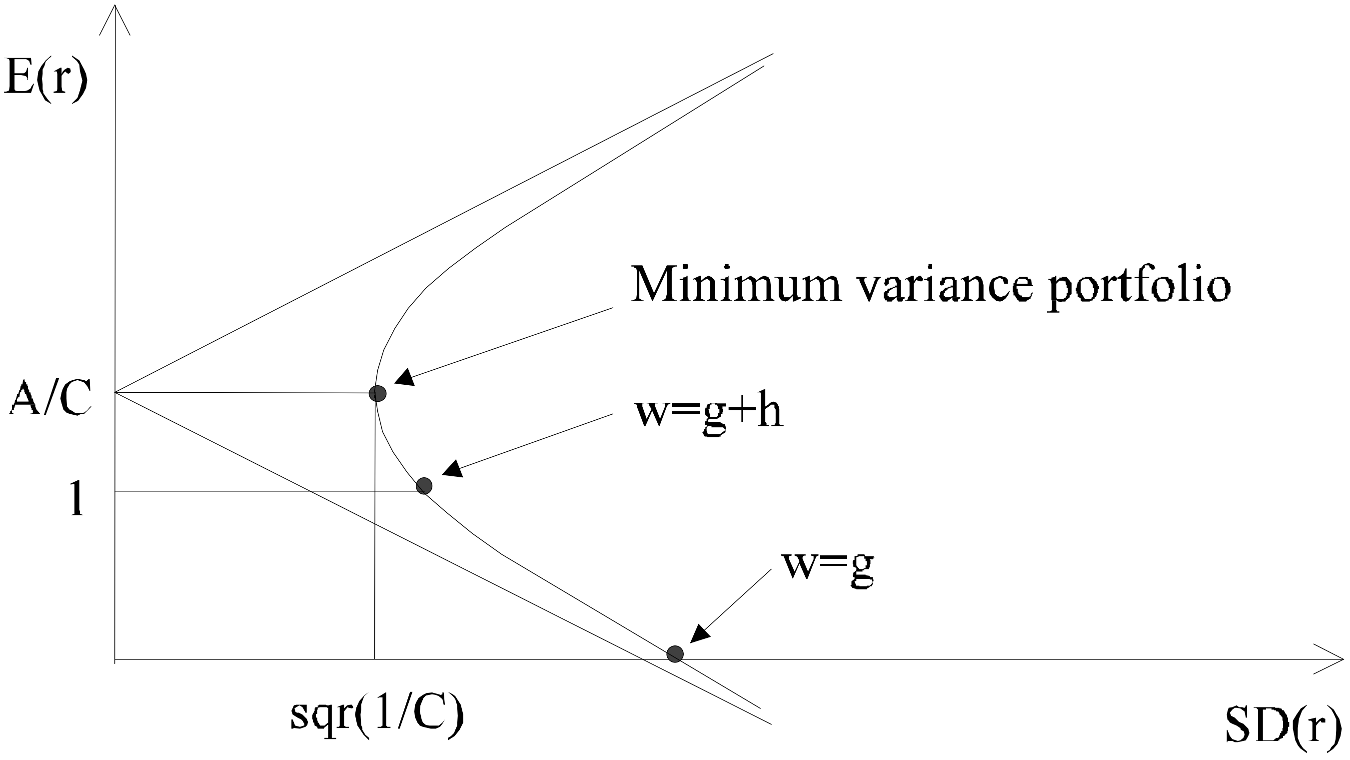
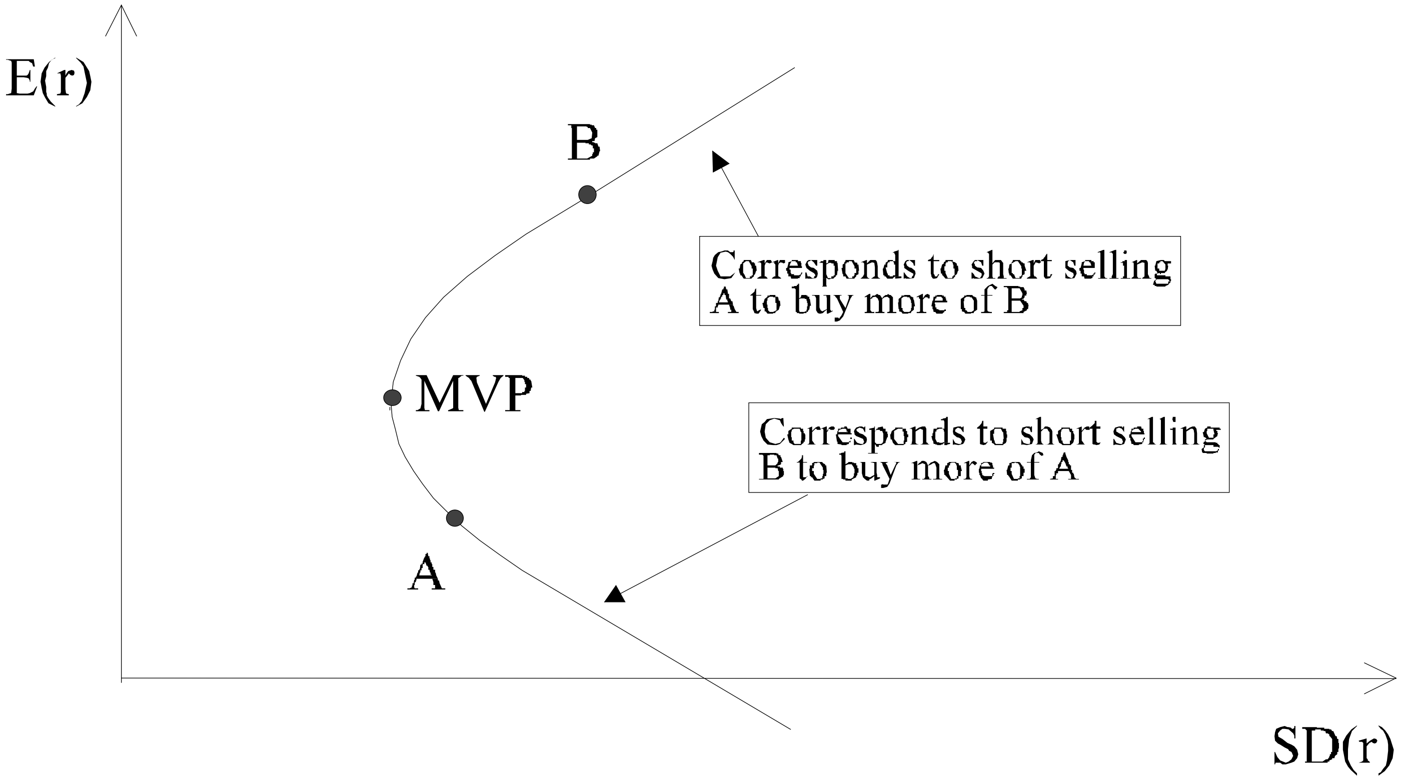


Figure 6-4 The Set of Frontier Portfolios: Mean/SD Space

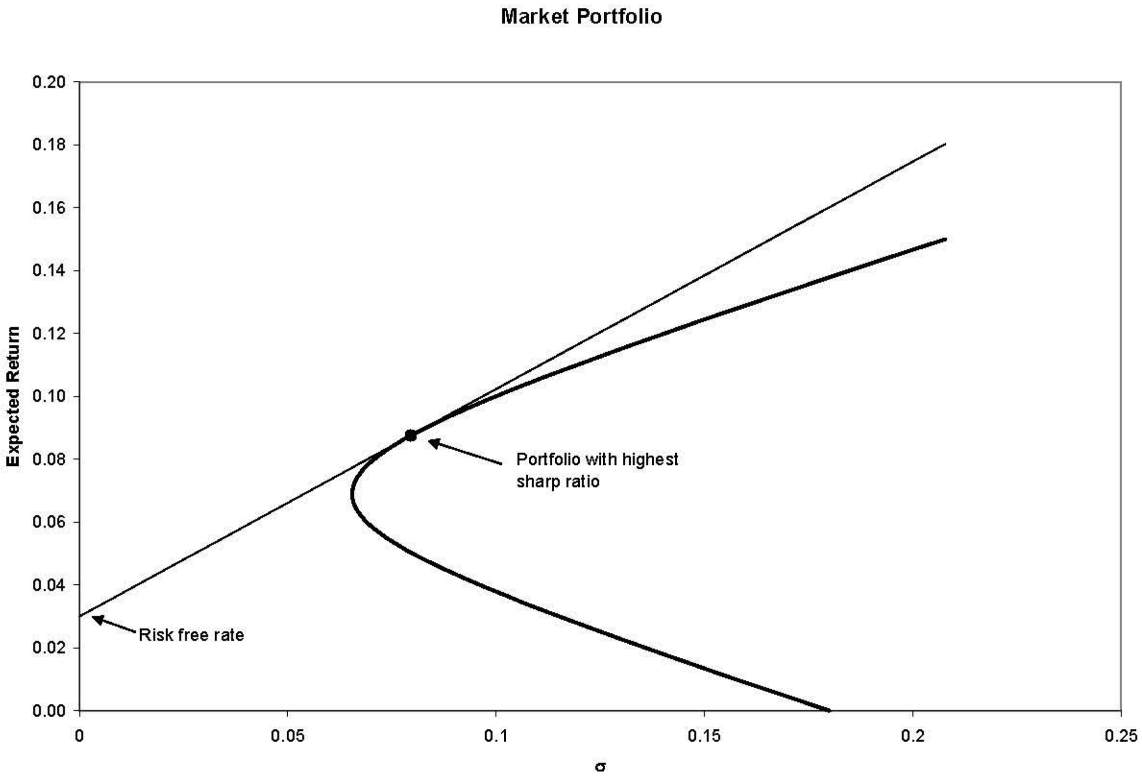


Corresponds to short selling A to buy more of B

Corresponds to short selling B to buy more of A

Figure 6-5 The Set of Frontier Portfolios: Short Selling Allowed

Efficient Frontier with risk-free asset



The Efficient Frontier: One Risk Free and n Risky Assets

Efficient Frontier with risk-free asset

- $\min_{\boldsymbol{w}} \frac{1}{2} \boldsymbol{w}' V \boldsymbol{w}$
  - s.t. $\boldsymbol{w}'\boldsymbol{\mu} + (1 - w^T \mathbf{1}) r^f = \mu^h$
  - FOC
    - $\boldsymbol{w}_h = \lambda V^{-1}\left(\boldsymbol{\mu} - r^f \mathbf{1}\right)$
    - Multiplying by $\left(\boldsymbol{\mu} - r^f \mathbf{1}\right)^T$ yields $\lambda = \frac{\mu^h - r^f}{\left(\boldsymbol{\mu} - r^f \mathbf{1}\right)' V^{-1}\left(\boldsymbol{\mu} - r^f \mathbf{1}\right)}$
  - Solution
    - $\boldsymbol{w}_h = \frac{V^{-1}(\boldsymbol{\mu} - r^f \mathbf{1})(\mu^h - r^f)}{H^2}$, where $H = \sqrt{B - 2Ar^f + C(r^f)^2}$

Efficient frontier with risk-free asset

- Result 1: Excess return in frontier excess return

$$\operatorname{cov}[r_h, r_p] = \boldsymbol{w}_h' V \boldsymbol{w}_p = \boldsymbol{w}_h'(\boldsymbol{\mu} - r^f \mathbf{1}) \frac{E[r_p] - r^f}{H^2}$$

$$= \frac{(E[r_h] - r^f)(E[r_p] - r^f)}{H^2}$$

$$\operatorname{var}[r_p] = \frac{(E[r_p] - r^f)^2}{H^2}$$

$$E[r_h] - r^f = \underbrace{\frac{\operatorname{cov}[r_h, r_p]}{\operatorname{var}[r_p]}}_{\beta_{h,p}} (E[r_p] - r^f)$$

(Holds for any frontier portfolio p, in particular the market portfolio)

Efficient Frontier with risk-free asset

- Result 2: Frontier is linear in $(E[r], \sigma)$-space

$$\mathrm{var}[r_h] = \frac{\left(E[r_h] - r_f\right)^2}{H^2}$$

$$E[r_h] = r_f + H\sigma_h$$

where H is the Sharpe ratio

$$H = \frac{E[r_h] - r_f}{\sigma_h}$$

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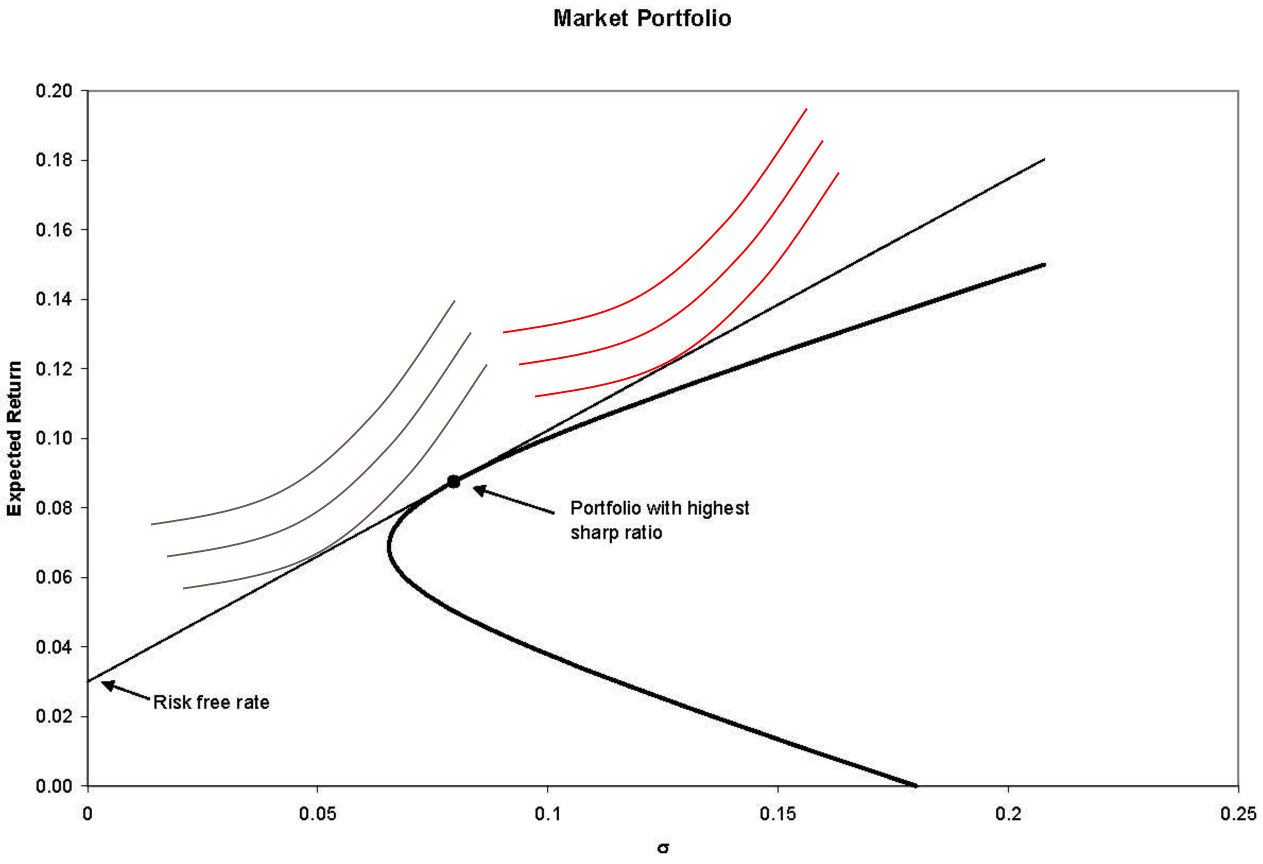
Aggregation: Two Fund Separation

- Doing it in two steps:
  - First solve frontier for *n* risky asset
  - Then solve tangency point
- Advantage:
  - Same portfolio of n risky asset for different agents with different risk aversion
  - Useful for applying equilibrium argument (later)

Recall HARA class of preferences

Two Fund Separation



Price of Risk =
= highest
Sharpe ratio

Optimal Portfolios of Two Investors with Different Risk Aversion

Mean-Variance Preferences

- $U(\mu_h, \sigma_h)$ with $\frac{\partial U}{\partial \mu_h} > 0, \frac{\partial U}{\partial \sigma_h^2} < 0$
  - Example: $E[W] - \frac{\rho}{2}\text{var}[W]$

- Also in expected utility framework
  - Example 1: Quadratic utility function (with portfolio return R)
    - $U(R) = a + bR + cR^2$
    - vNM: $E[U(R)] = a + bE[R] + cE[R^2] = a + b\mu_h + c\mu_h^2 + c\sigma_h^2 = g(\mu_h, \sigma_h)$
  - Example 2: CARA Gaussian
    - asset returns jointly normal ⇒ $\sum_i w^i r^i$ normal
    - If U is CARA ⇒ certainty equivalent is $\mu_h - \frac{\rho}{2}\ \sigma_h^2$

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Equilibrium leads to CAPM

- Portfolio theory: only analysis of demand
  - price/returns are taken as given
  - composition of risky portfolio is same for all investors

- Equilibrium Demand = Supply (market portfolio)

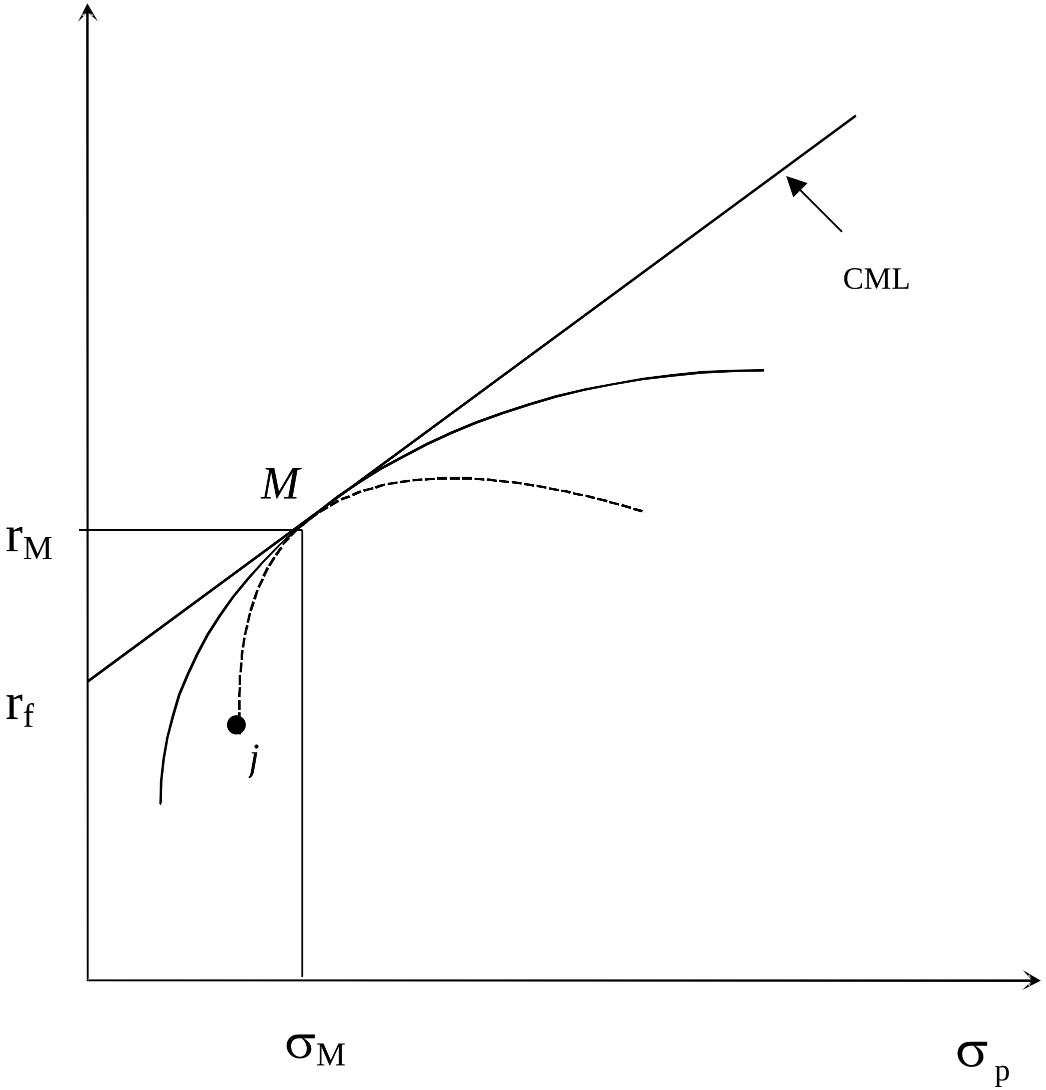
- CAPM allows to derive
  - equilibrium prices/ returns.
  - risk-premium

The CAPM with a risk-free bond

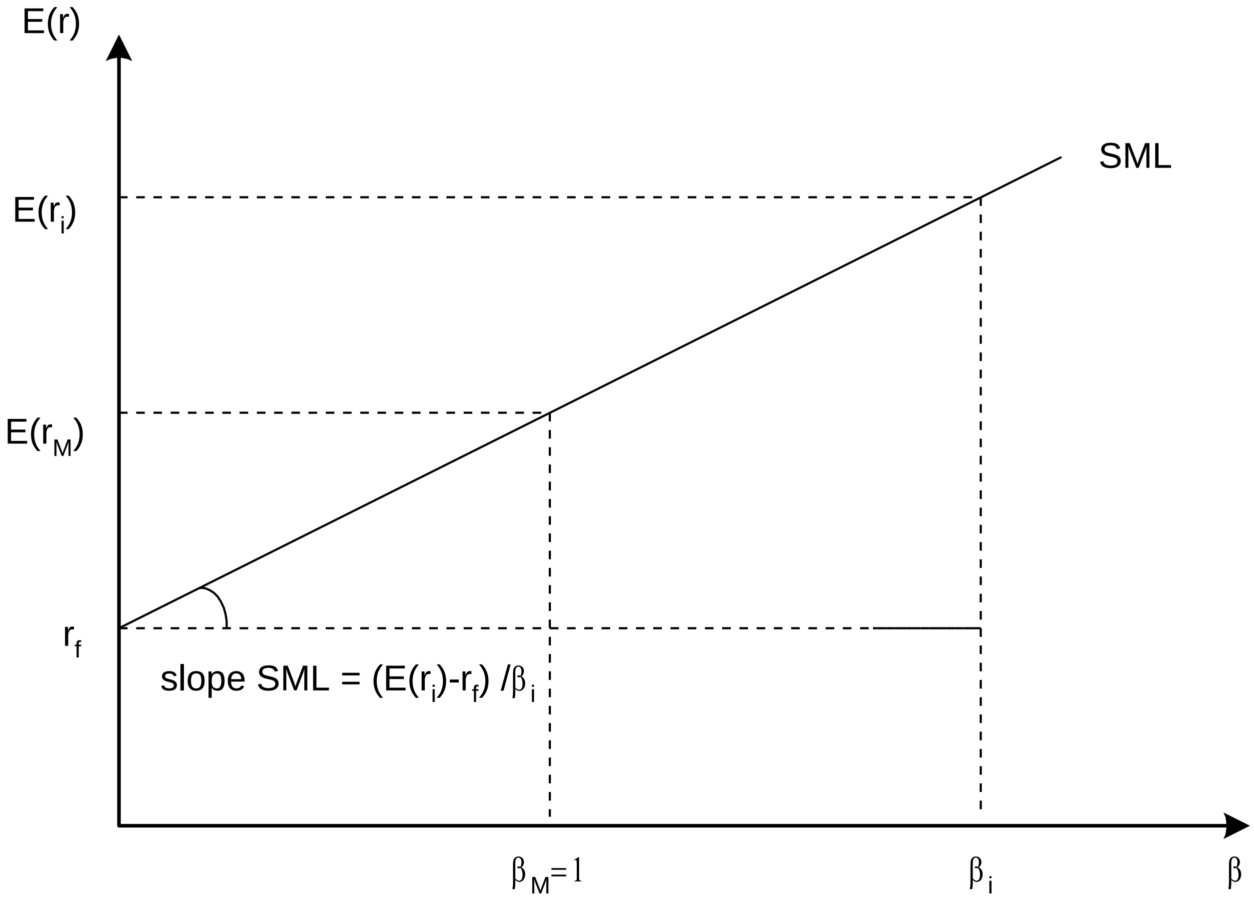
- The market portfolio is efficient since it is on the efficient frontier.
- All individual optimal portfolios are located on the half-line originating at point $(0, rf)$.
- The slope of Capital Market Line (CML): $\frac{E[R^{mkt}]-R_f}{\sigma_{mkt}}$

$$E[R_h] = R_f + \frac{E[R^{mkt}] - R_f}{\sigma_{mkt}} \sigma_h$$

The Capital Market Line



The Security Market Line



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Projections

- States $s = 1, \dots, S$ with $\pi_s > 0$
- Probability inner product

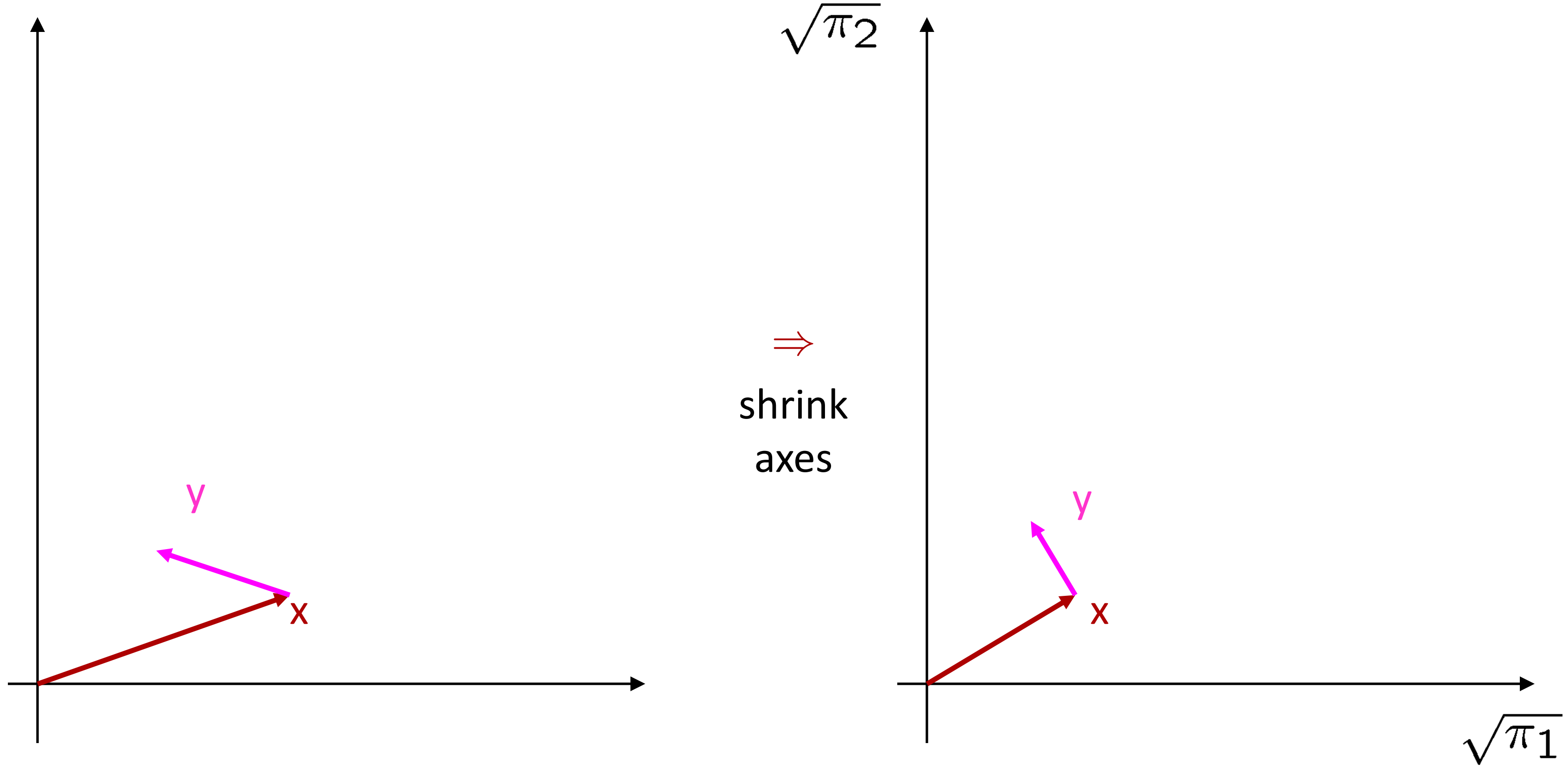
$$[x, y]_\pi = \sum_s \pi_s x_s y_s = \sum_s \sqrt{\pi_s} x_s \sqrt{\pi_s} y_s$$

- π-norm $\|x\| = \sqrt{[x, x]_\pi}$ (measure of length)

i. $\|x\| > 0 \ \ \forall x \neq 0$ and $\|x\| = 0$ if $x = 0$

ii. $\|\lambda x\| = |\lambda| \|x\|$

iii. $\|x + y\| \leq \|x\| + \|y\| \ \ \forall x; y \in \mathbb{R}^S$



x and y are π-orthogonal iff $[x, y]_\pi = 0$, i.e. $E[xy] = 0$

…Projections…

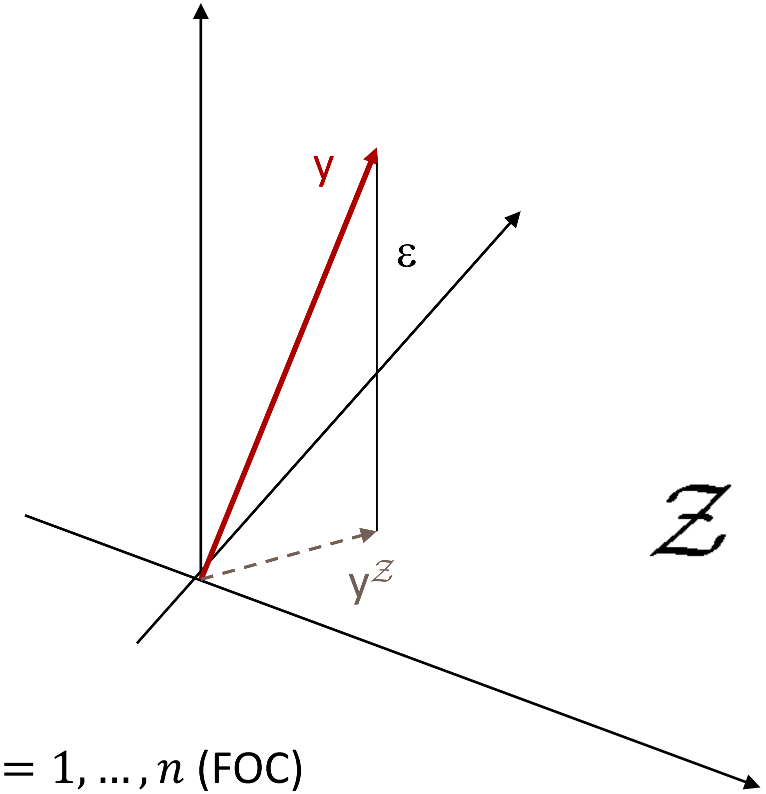
- $\mathcal{Z}$ space of all linear combinations of vectors $z_1, \dots, z_n$
- Given a vector $y \in \mathbb{R}^S$ solve

$$\min_{\alpha \in \mathbb{R}^n} E\left[y - \sum_j \alpha^j z^j\right]^2$$

- FOC: $\sum_s \pi_s \left(y_s - \sum_j \alpha^j z_s^j\right) z^j = 0$
  - Solution $\hat{\alpha} \Rightarrow y^{\mathcal{Z}} = \sum_j \hat{\alpha}^j z^j, \epsilon := y - y^{\mathcal{Z}}$

- [smallest distance between vector y and $\mathcal{Z}$ space]

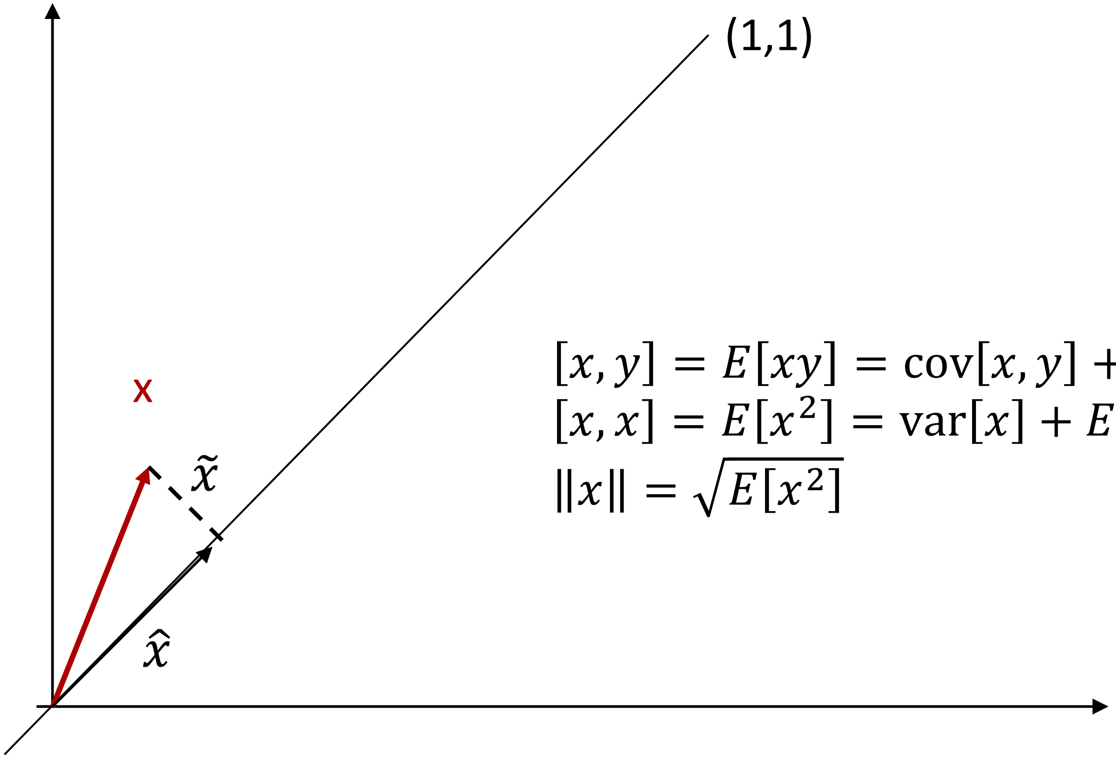
…Projections



$E\left[\varepsilon z^j\right] = 0$ for each $j = 1, \dots, n$ (FOC)
$\varepsilon \perp z$
y^z is the (orthogonal) projection on $\mathcal{Z}$
$y = y^{\mathcal{Z}} + \varepsilon', y^{\mathcal{Z}} \in \mathcal{Z}, \varepsilon \perp z$

Expected Value and Co-Variance…

squeeze axis by $\sqrt{\pi_s}$



$$[x,y] = E[xy] = \text{cov}[x,y] + E[x]E[y]$$
$$[x,x] = E[x^2] = \text{var}[x] + E[x]^2$$
$$\|x\| = \sqrt{E[x^2]}$$

$$x = \hat{x} + \tilde{x}$$

…Expected Value and Co-Variance

- $x = \hat{x} + \tilde{x}$ where
  - – $\hat{x}$ is a projection of x onto $\langle 1 \rangle$
  - – $\tilde{x}$ is a projection of x onto $\langle 1 \rangle^{\perp}$
- $E[x] = [x, 1]_{\pi} = [\hat{x}, 1]_{\pi} = \hat{x}[1,1]_{\pi} = \hat{x}$

  scalar
  slight abuse of notation
- $\mathrm{var}[x] = [\tilde{x}, \tilde{x}]_{\pi} = \mathrm{var}[\tilde{x}]$
  - – $\sigma_x = \|\tilde{x}\|_{\pi}$
- $\mathrm{cov}[x, y] = \mathrm{cov}[\tilde{x}, \tilde{y}] = [\tilde{x}, \tilde{y}]_{\pi}$
- Proof: $[x, y]_{\pi} = [\hat{x}, \hat{y}]_{\pi} + [\tilde{x}, \tilde{y}]_{\pi}$
  - – $[\hat{y}, \tilde{x}]_{\pi} = [\tilde{y}, \hat{x}]_{\pi} = 0, [x, y]_{\pi} = E[\hat{y}]E[\hat{x}] + \mathrm{cov}[\tilde{x}, \tilde{y}]$

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for given prices/returns

Pricing Kernel m^*…

- $\langle X \rangle$ space of feasible payoffs.
- If no arbitrage and $\pi \gg 0$ there exists SDF $m \in \mathbb{R}^S, m \gg 0$, such that $q(z) = E[mz]$.
- $m \in \mathbb{R}^S$– SDF need not be in asset span.
- A pricing kernel is a $m^* \in \langle X \rangle$ such that for each $z \in \langle X \rangle, q(z) = E[m^* z]$

…Pricing Kernel - Examples…

- Example 1:
  - $S = 3, \pi^s = \frac{1}{3}$
  - $x_1 = (1,0,0), x_2 = (0,1,1)$ and $p = \left(\frac{1}{3}, \frac{2}{3}\right)$
  - Then $m^* = (1,1,1)$ is the unique pricing kernel.
- Example 2:
  - $x_1 = (1,0,0), x_2 = (0,1,0), p = \left(\frac{1}{3}, \frac{2}{3}\right)$
  - Then $m^* = (1,2,0)$ is the unique pricing kernel.

…Pricing Kernel – Uniqueness

- If a state price density exists, there exists a ***unique*** pricing kernel.
  - If $\dim\langle X\rangle = S$ (markets are complete),
    there are exactly m equations and m unknowns
  - If $\dim\langle X\rangle < S$, (markets may be incomplete)
    For any state price density (=SDF) m and any $z \in \langle X\rangle$
    $$E[(m - m^*)z] = 0$$
    $m = (m - m^*) + m^* \Rightarrow m^*$is the **“projection”** of m on $\langle X\rangle$
    - Complete markets $\Rightarrow m^* = m$ (SDF=state price density)

Expectations Kernel k^*

- An expectations kernel is a vector $k^* \in \langle X \rangle$
  - Such that $E[z] = E[k^* z]$ for each $z \in \langle X \rangle$
- Example
  - $S = 3, \pi^s = \frac{1}{3}, x_1 = (1,0,0), x_2 = (0,1,0)$
  - Then the unique $k^* = (1,1,0)$
- If $\pi \gg 0$, there exists a unique expectations kernel.
- Let $I = (1, \dots, 1)$ then for any $z \in \langle X \rangle$

$$E[(I - k^*)z] = 0$$

  - $k^*$is the **“projection”** of I on $\langle X \rangle$
  - $k^* = I$ if bond can be replicated (e.g. if markets are complete)

Mean Variance Frontier

- *Definition 1:* $z \in \langle X \rangle$ is in the mean variance frontier if there exists no $z' \in \langle X \rangle$ such that $E[z'] = E[z], q(z') = q(z)$ and $\mathrm{var}[z'] < \mathrm{var}[z]$

- *Definition 2:* Let $\mathcal{E}$ be the space generated by $m^*$and k^*
  - Decompose $z = z^{\varepsilon} + \varepsilon$ with $z^{\mathcal{E}} \in \mathcal{E}$ and $\varepsilon \perp \mathcal{E}$
  - Hence, $E[\varepsilon] = E[\varepsilon k^*] = 0, q(\varepsilon) = E[\varepsilon m^*] = 0$
    $\mathrm{cov}[\varepsilon, z^{\varepsilon}] = E[\varepsilon z^{\varepsilon}] = 0$, since $\varepsilon \perp \mathcal{E}$
  - $\mathrm{var}[z] = \mathrm{var}[z^{\varepsilon}] + \mathrm{var}[\varepsilon]$ (price of ε is zero, but positive variance)

- z is in mean variance frontier $\Rightarrow$ z $\in \mathcal{E}$.
  - Every $z \in \mathcal{E}$ is in mean variance frontier.

Frontier Returns…

- Frontier returns are the returns of frontier payoffs with non-zero prices.

[Note: R indicates Gross return]

$$R_{k^*} = \frac{k^*}{q(k^*)} = \frac{k^*}{E[m^*]}$$

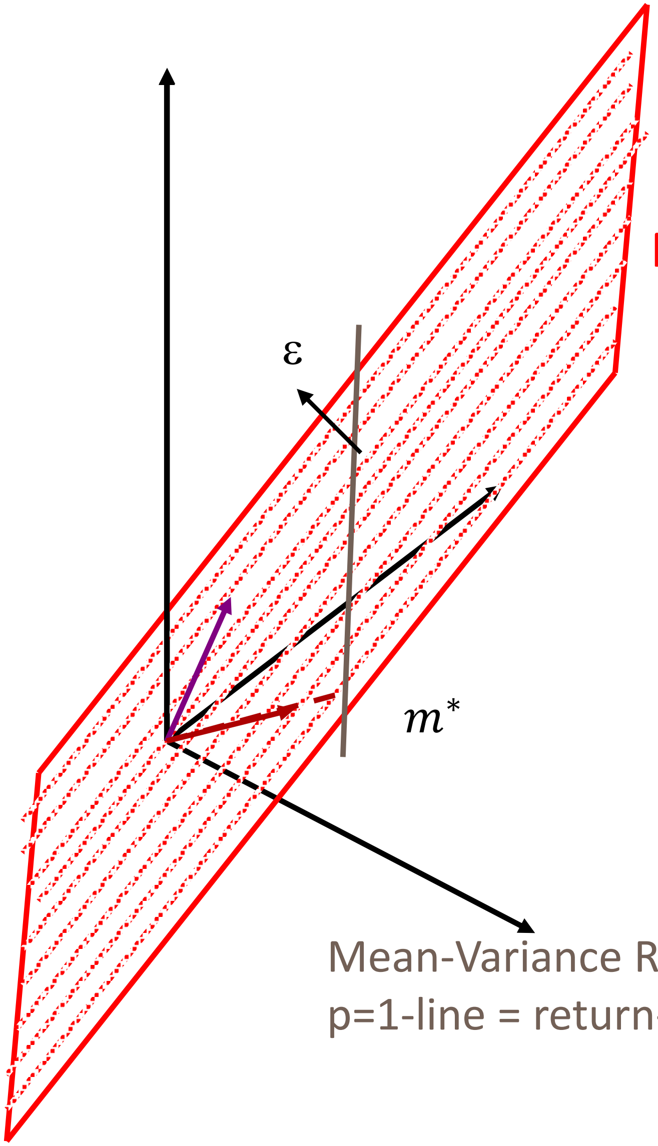
$$R_{m^*} = \frac{m^*}{q(m^*)} = \frac{m^*}{E[m^* m^*]}$$

- If $z = \alpha m^* + \beta k^*$ then

$$R_z = \underbrace{\frac{\alpha q(m^*)}{\alpha q(m^*) + \beta q(k^*)}}_{\lambda} R_{m^*} + \underbrace{\frac{\beta q(k^*)}{\alpha q(m^*) + \beta q(k^*)}}_{1-\lambda} R_{k^*}$$

- graphically: payoffs with price of p=1.

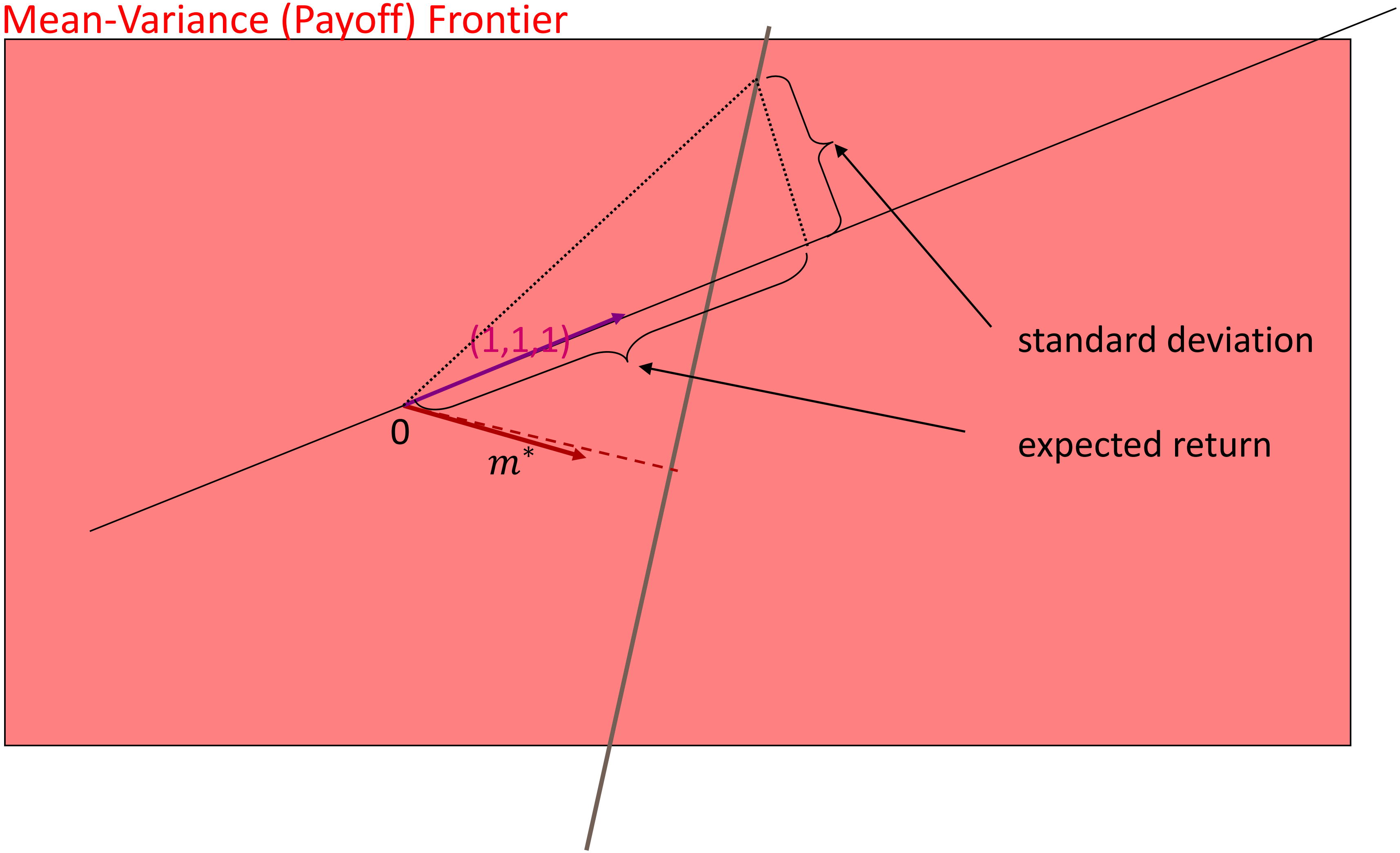
$\langle X \rangle$ = R^S = R^3



Mean-Variance Payoff Frontier

Mean-Variance Return Frontier
p=1-line = return-line (orthogonal to m^*)

Mean-Variance (Payoff) Frontier

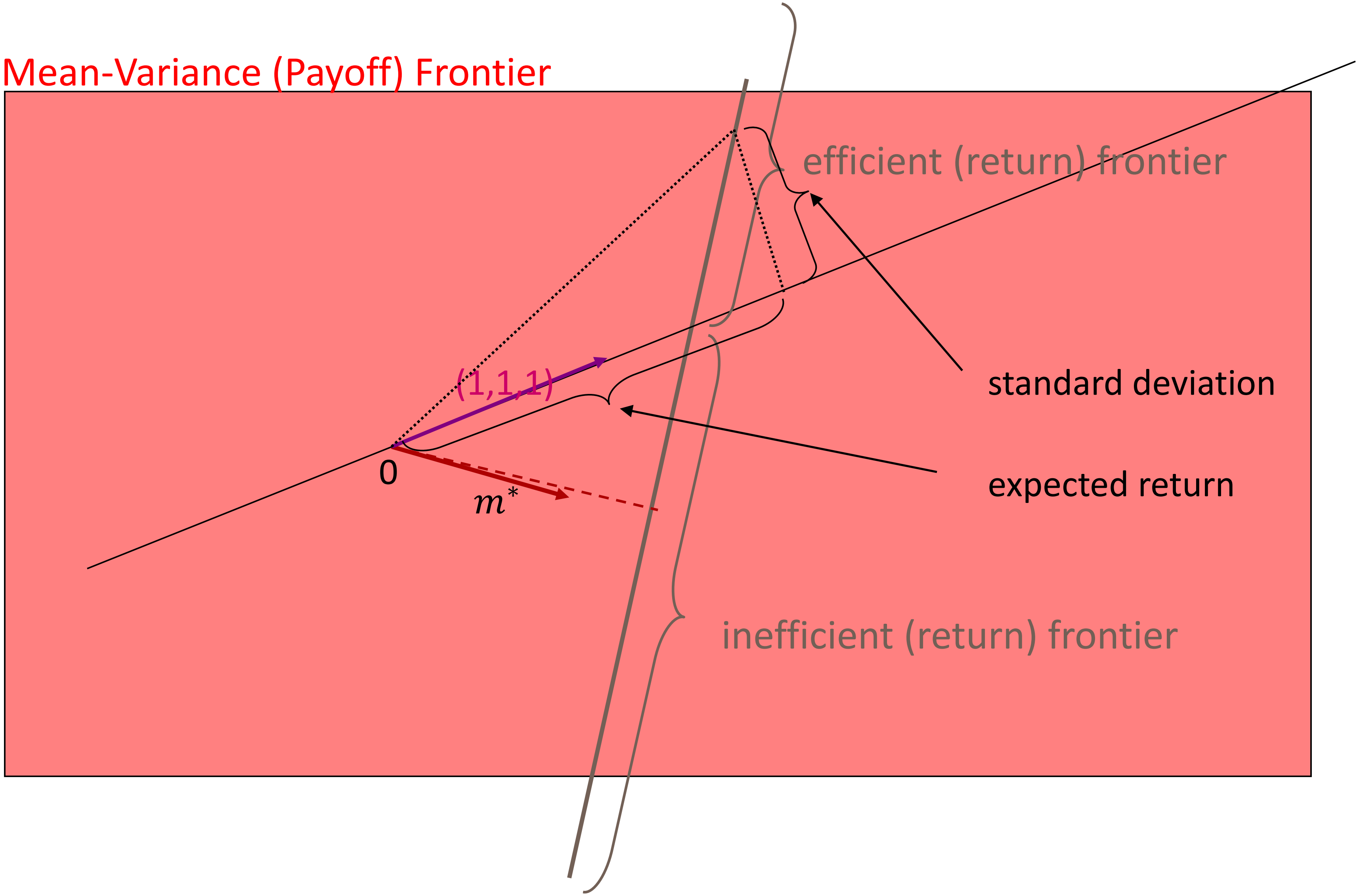


standard deviation

expected return

NB: graphical illustrated of expected returns and standard deviation changes if bond is not in payoff span.

Mean-Variance (Payoff) Frontier



…Frontier Returns

(if agent is risk-neutral)

- If $k^* = \alpha m^*$, frontier returns $\equiv R_{k^*}$
- If$k^* \neq \alpha m^*$, frontier returns can be written as:

$$R_\lambda = R_{k^*} + \lambda(R_{m^*} - R_{k^*})$$

- Expectations and variance are

$$E[R_\lambda] = E[R_{k^*}] + \lambda(E[R_{m^*}] - E[R_{k^*}])$$

$$\text{var}[R_\lambda] =$$
$$= \text{var}[R_{k^*}] + 2\lambda\text{cov}[R_{k^*}, R_{m^*} - R_{k^*}] + \lambda^2\text{var}[R_{m^*} - R_{k^*}]$$

- If risk-free asset exists, these simplify to:

$$E[R_\lambda] = R_f + \lambda\left(E[R_{m^*}] - R_f\right) = R_f \pm \sigma(R_\lambda)\frac{E[R_{m^*}] - R_f}{\sigma(R_{m^*})}$$

$$\text{var}[R_\lambda] = \lambda^2\text{var}[R_{m^*}], \sigma(R_\lambda) = |\lambda|\sigma(R_{m^*})$$

Minimum Variance Portfolio

- Take FOC w.r.t. λ of

$$\begin{aligned}&\operatorname{var}[R_\lambda]\\&= \operatorname{var}[R_{k^*}] + 2\lambda \operatorname{cov}[R_{k^*}, R_{m^*} - R_{k^*}]\\&+ \lambda^2 \operatorname{var}[R_{m^*} - R_{k^*}]\end{aligned}$$

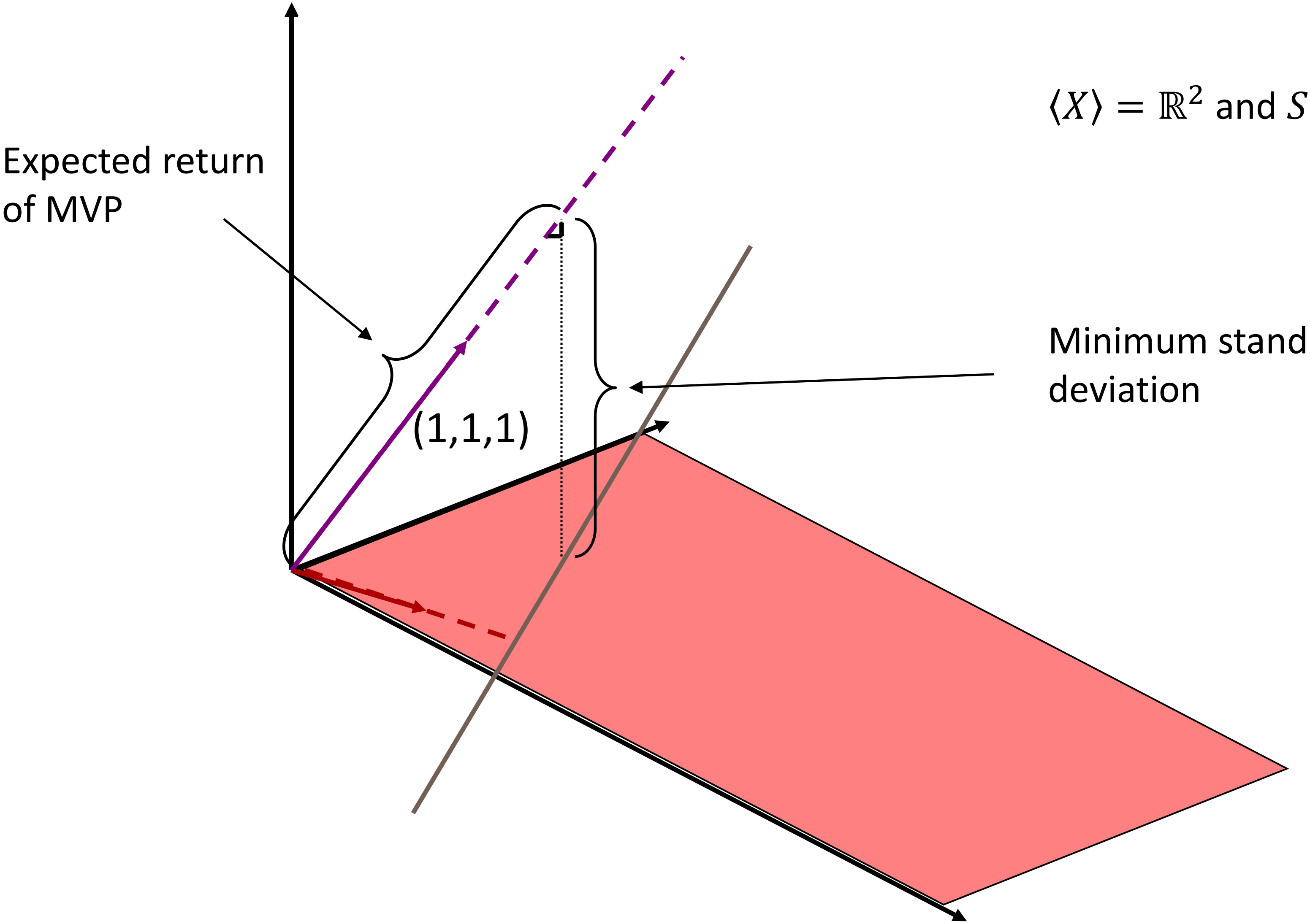
- Hence, MVP has return of

$$R_{k^*} + \lambda_0 (R_{m^*} - R_{k^*})$$

$$\lambda_0 = -\frac{\operatorname{cov}[R_{k^*}, R_{m^*} - R_{k^*}]}{\operatorname{var}[R_{m^*} - R_{k^*}]}$$

Illustration of MVP

$\langle X \rangle = \mathbb{R}^2$ and $S = 3$



Mean-Variance Efficient Returns

- *Definition:* A return is **mean-variance efficient** if there is no other return with same variance but greater expectation.
- Mean variance efficient returns are frontier returns with $E[R_\lambda] \geq E\left[R_{\lambda_0}\right]$

- If risk-free asset can be replicated
  - Mean variance efficient returns correspond to λ_0.
  - Pricing kernel (portfolio) is not mean-variance efficient, since $E[R_{m^*}] = \frac{E[m^*]}{E[(m^*)^2]} < \frac{1}{E[m^*]} = R_f$

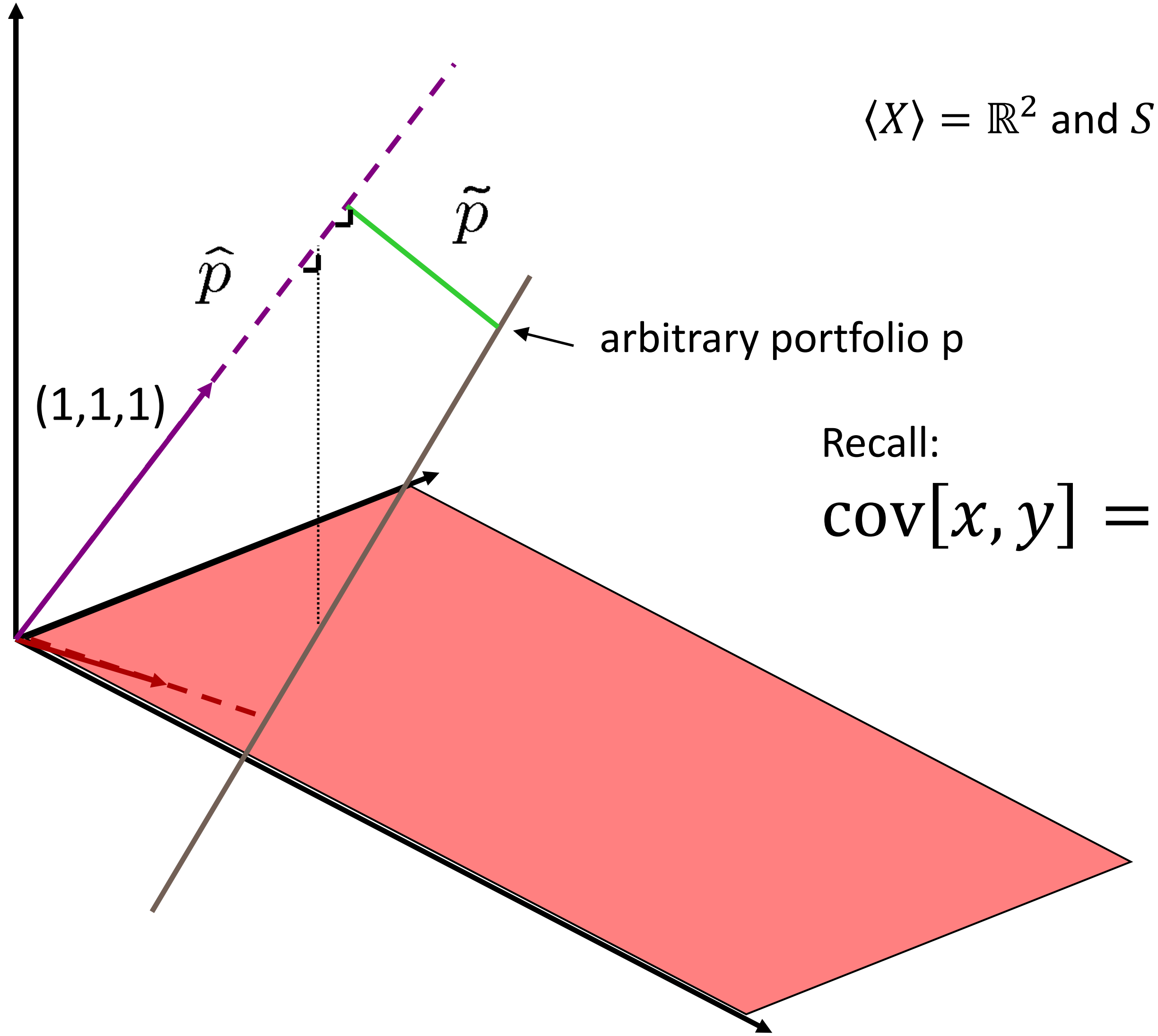
Zero-Covariance Frontier Returns

- Take two frontier portfolios with returns $R_\lambda = R_{k^*} + \lambda(R_{m^*} - R_{k^*})$ and $R_\mu = R_{k^*} + \mu(R_{m^*} - R_{k^*})$
- $\mathrm{cov}\left[R_\mu, R_\lambda\right] = \mathrm{var}[R_{k^*}] + (\lambda + \mu)\mathrm{cov}[R_{k^*}, R_{m^*} - R_{k^*}] + \lambda\mu\mathrm{var}[R_{m^*} - R_{k^*}]$
- The portfolios have zero co-variance if

$$\mu = -\frac{\mathrm{var}[R_{k^*}] + \lambda\mathrm{cov}[R_{k^*}, R_{m^*} - R_{k^*}]}{\mathrm{cov}[R_{k^*}, R_{m^*} - R_{k^*}] + \lambda\mathrm{var}\,[R_{m^*} - R_{k^*}]}$$

- For all $\lambda \neq \lambda_0$, μ exists
  - $\mu = 0$ if risk-free bond can be replicated

Illustration of ZC Portfolio…

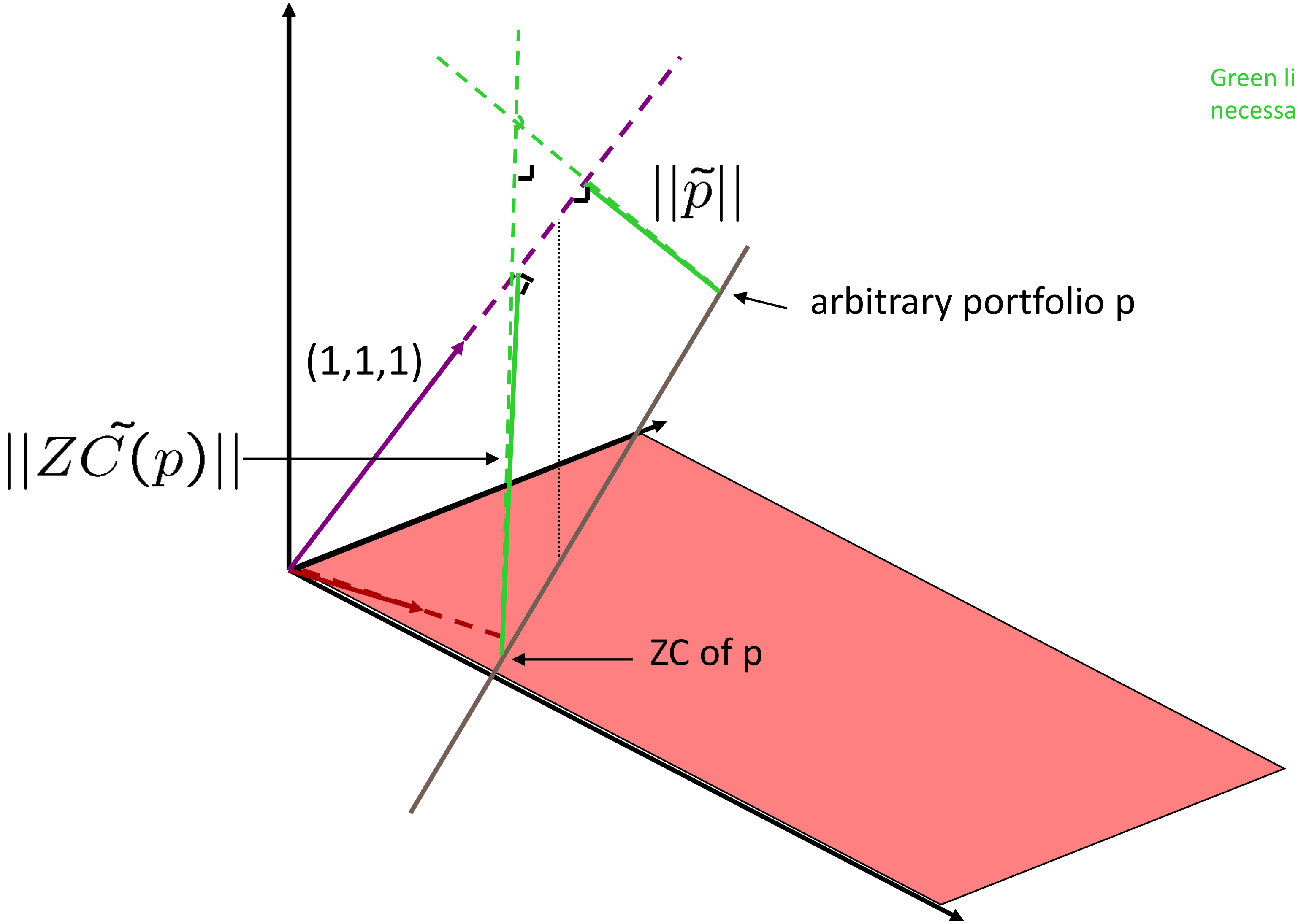


$\langle X \rangle = \mathbb{R}^2$ and $S = 3$

Recall:

$$\text{cov}[x, y] = [\tilde{x}, \tilde{y}]_\pi$$

…Illustration of ZC Portfolio



Green lines do not necessarily cross.

Beta Pricing…

- Frontier Returns (are on linear subspace). Hence

$$R_\beta = R_\mu + \beta\left(R_\lambda - R_\mu\right)$$

- Consider any asset with payoff x_j
  - It can be decomposed in $x_j = x_j^\varepsilon + \varepsilon_i$
  - $q\left(x_j\right) = q\left(x_j^\varepsilon\right)$ and $E\left[x_j\right] = E\left[x_j^\varepsilon\right]$, since $\varepsilon \perp \mathcal{E}$
  - Return of x_j is $R_j = R_j^\varepsilon + \frac{\varepsilon_j}{q(x_j)}$
  - Using above and assuming $\lambda \neq \lambda_0$ and μ is ZC-portfolio of λ,

$$R_j = R_\mu + \beta_j\left(R_\lambda - R_\mu\right) + \frac{\varepsilon_j}{q\left(x_j\right)}$$

…Beta Pricing

- Taking expectations and deriving covariance
- $E[R_j] = E[R_\mu] + \beta_j (E[R_\lambda] - E[R_\mu])$
- $\text{cov}[R_\lambda, R_j] = \beta_j \text{var}[R_\lambda] \Rightarrow \beta_j = \frac{\text{cov}[R_\lambda, R_j]}{\text{var}[R_\lambda]}$
  - Since $R_\lambda \perp \frac{\varepsilon_j}{q(x_j)}$
- If risk-free asset can be replicated, beta-pricing equation simplifies to

$$E[R_j] = R_f + \beta_j (E[R_\lambda] - R_f)$$

- Problem: How to identify frontier returns

Capital Asset Pricing Model…

- CAPM = market return is frontier return
  - Derive conditions under which market return is frontier return
  - Two periods: 0,1.
  - Endowment: individual w_1^i at time 1, aggregate $\overline{w}_1 = \overline{w}_1^{\langle X\rangle} + \overline{w}_1^{\langle Y\rangle}$, where $\overline{w}_1^{\langle X\rangle}, \overline{w}_1^{\langle Y\rangle}$ are orthogonal and $\overline{w}_1^{\langle X\rangle}$ is the orthogonal projection of $\overline{w}_1$ on $\langle X\rangle$.
  - The market payoff is $\overline{w}_1^{\langle X\rangle}$
  - Assume $q\left(\overline{w}_1^{\langle X\rangle}\right) \neq 0$, let $R_{mkt} = \frac{\overline{w}_1^{\langle X\rangle}}{q\left(\overline{w}_1^{\langle X\rangle}\right)}$, and assume that R_{mkt} is not the minimum variance return.

…Capital Asset Pricing Model

- If R_0 is the frontier return that has zero covariance with R_{mkt} then, for every security j,
- $E\left[R_j\right] = E[R_0] + \beta_j(E[R_{mkt}] - E[R_0])$ with $\beta_j = \frac{\text{cov}[R_j, R_{mkt}]}{\text{var}[R_{mkt}]}$

- If a risk free asset exists, equation becomes, $E[R_j] = R_f + \beta_j\left(E[R_{mkt}] - R_f\right)$

- N.B. first equation always hold if there are only two assets.

Overview

1. Introduction:
   Simple CAPM with quadratic utility functions
2. Traditional Derivation of CAPM
   - Demand: Portfolio Theory
   - Aggregation: Fund Separation Theorem
   - Equilibrium: CAPM

   } for given prices/returns
3. Modern Derivation of CAPM
   - Projections
   - Pricing Kernel and Expectation Kernel
4. Testing CAPM
5. Practical Issues – Black-Litterman

Practical Issues

- Testing of CAPM
- Jumping weights
  - Domestic investments
  - International investment
- Black-Litterman solution

Testing the CAPM

- Take CAPM as given and test empirical implications

- Time series approach
  - Regress individual returns on market returns
$$R_{it} - R_{ft} = \hat{\alpha}_i + \hat{\beta}_{im}\left(R_{mt} - R_{ft}\right) + \varepsilon_{it}$$
  - Test whether **constant term** $\alpha_i = 0$

- Cross sectional approach
  - Estimate betas from time series regression
  - Regress individual returns on betas
$$R_i = \lambda\hat{\beta}_{im} + \alpha_i$$
  - Test whether **regression residuals** $\alpha_i = 0$

Empirical Evidence

- Excess returns on high-beta stocks are low
- Excess returns are high for small stocks
  - Effect has been weak since early 1980s
- Value stocks have high returns despite low betas
- Momentum stocks have high returns and low betas

Reactions and Critiques

- Roll Critique
  - The CAPM is not testable because composition of true market portfolio is not observable
- Hansen-Richard Critique
  - The CAPM could hold *conditionally* at each point in time, but fail unconditionally
- Anomalies are result of “data mining”
- Anomalies are concentrated in small, illiquid stocks
- Markets are inefficient – “joint hypothesis test”

Practical Issues

- Estimation
  - How do we estimate all the parameters we need for portfolio optimization?
- What is the market portfolio?
  - Restricted short-sales and other restrictions
  - International assets & currency risk
- How does the market portfolio change over time?
  - Empirical evidence
  - More in dynamic models

Overview

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MV Portfolio Selection in Real Life

- An investor seeking to use mean-variance portfolio construction has to
  - Estimate N means,
  - N variances,
  - N*(N-1)/2 co-variances

- <u>Estimating means</u>
  - For any partition of [0,T] with N points (Δt=T/N):

$$E[r] = \frac{1}{\Delta t} \cdot \left(\frac{1}{N} \cdot \sum_{i=1}^{N} r_{i \cdot \Delta t} \right) = \frac{p_T - p_0}{T} \text{ (in log prices)}$$

  - Knowing the first and last price is sufficient

Estimating Means

- Let X_k denote the logarithmic return on the market, with $k = 1, \dots, n$ over a period of length h
  - The dynamics to be estimated are:

$$X_k = \mu \cdot \Delta + \sigma \cdot \sqrt{\Delta} \cdot \epsilon_k$$

where the ϵ_k are i.i.d. standard normal random variables.

  - The standard estimator for the expected logarithmic mean rate of return is:

$$\hat{\mu} = \frac{1}{h} \cdot \sum_1^n X_k$$

where
h is length of observation
n number of observations
$\Delta = n/h$

  - The mean and variance of this estimator

$$E[\hat{\mu}] = \frac{1}{h} \cdot E\left[\sum_1^n X_k\right] = \frac{1}{h} \cdot n \cdot \mu \cdot \Delta = \mu$$

$$Var[\hat{\mu}] = \frac{1}{h^2} \cdot Var\left[\sum_1^n X_k\right] = \frac{1}{h^2} \cdot n \cdot \sigma^2 \cdot \Delta = \frac{\sigma^2}{h}$$

  - The accuracy of the estimator depends only upon the total length of the observation period (*h*), and *not* upon the number of observations (*n*).

Estimating Variances

- Consider the following estimator:

$$\widehat{\sigma^2} = \frac{1}{h} \cdot \sum_{i=1}^{n} X_k^2$$

- The mean and variance of this estimator:

$$E\left[\widehat{\sigma^2}\right] = \frac{1}{h} \cdot \sum_{i=1}^{n} E\left[X_k^2\right] = \frac{1}{h} \cdot n \cdot \left(\mu^2 \cdot \Delta^2 + \sigma^2 \cdot \Delta\right) = \sigma^2 + \mu^2 \cdot \frac{h}{n}$$

$$Var\left[\widehat{\sigma^2}\right] = \frac{1}{h^2} \cdot Var\left[\sum_{i=1}^{n} X_k^2\right] = \frac{1}{h^2} \cdot \sum_{i=1}^{n} Var\left[X_k^2\right] = \frac{n}{h^2} \cdot \left(E\left[X_k^4\right] - E\left[X_k^2\right]^2\right) = \frac{2 \cdot \sigma^4}{n} + \frac{4 \cdot \mu^2 \cdot h}{n^2}$$

  - The estimator is biased b/c we did not subtract out the expected return from each realization.
  - Magnitude of the bias declines as *n* increases.
  - For a fixed *h*, the accuracy of the variance estimator can be improved by sampling the data more frequently.

Estimating variances: Theory vs. Practice

- For any partition of $[0, T]$ with N points ($\Delta t = T/N$):

$$Var[r] = \frac{1}{N} \cdot \sum_{i=1}^{N} (r_{i \cdot \Delta t} - E[r])^2 \rightarrow \sigma^2 \;\; as\ N \rightarrow \infty$$

- Theory: Observing the same time series at progressively higher frequencies increases the precision of the estimate.
- Practice:
  - Over shorter interval increments are non-Gaussian
  - Volatility is time-varying (GARCH, SV-models)
  - Market microstructure noise

Estimating covariances: Theory vs. Practice

- In theory, the estimation of covariances shares the features of variance estimation.
- In practice:
  - Difficult to obtain synchronously observed time-series -> may require interpolation, which affects the covariance estimates.
  - The number of covariances to be estimated grows very quickly, such that the resulting covariance matrices are unstable (check condition numbers!).
  - Shrinkage estimators (Ledoit and Wolf, 2003, “Honey, I Shrunk the Covariance Matrix”)

Unstable Portfolio Weights

- Are optimal weights statistically different from zero?
  - Properly designed regression yields portfolio weights
  - Statistical tests for significance of weight
- Example: Britton-Jones (1999) for international portfolio
  - Fully hedged USD Returns
    - Period: 1977-1966
    - 11 countries
  - Results
    - Weights vary significantly across time and in the cross section
    - Standard errors on coefficients tend to be large

Britton-Jones (1999)

	1977-1996		1977-1986		1987-1996	
	weights	t-stats	weights	t-stats	weights	t-stats
Australia	12.8	0.54	6.8	0.20	21.6	0.66
Austria	3.0	0.12	-9.7	-0.22	22.5	0.74
Belgium	29.0	0.83	7.1	0.15	66	1.21
Canada	-45.2	-1.16	-32.7	-0.64	-68.9	-1.10
Denmark	14.2	0.47	-29.6	-0.65	68.8	1.78
France	1.2	0.04	-0.7	-0.02	-22.8	-0.48
Germany	-18.2	-0.51	9.4	0.19	-58.6	-1.13
Italy	5.9	0.29	22.2	0.79	-15.3	-0.52
Japan	5.6	0.24	57.7	1.43	-24.5	-0.87
UK	32.5	1.01	42.5	0.99	3.5	0.07
US	59.3	1.26	27.0	0.41	107.9	1.53

Black-Litterman Appraoch

- Since portfolio weights are very unstable, we need to discipline our estimates somehow
  - Our current approach focuses only on historical data

- Priors
  - Unusually high (or low) past return may not (on average) earn the same high (or low) return going forward
  - Highly correlated sectors should have similar expected returns
  - A “good deal” in the past (i.e. a good realized return relative to risk) should not persist if everyone is applying mean-variance optimization.

- Black Litterman Approach
  - Begin with “CAPM prior”
  - Add views on assets or portfolios
  - Update estimates using Bayes rule

Black-Litterman Model: Priors

- Suppose the returns of N risky assets (in vector/matrix notation) are
$$r \sim \mathcal{N}(\mu, \Sigma)$$

- CAPM: The equilibrium risk premium on each asset is given by:
$$\Pi = \gamma \cdot \Sigma \cdot w_{eq}$$
  - γ is the investors coefficient of risk aversion.
  - w_{eq} are the equilibrium (i.e. market) portfolio weights.

- The investor is assumed to start with the following Bayesian prior (with imprecision):
$$\mu = \Pi + \epsilon^{eq} \quad \text{where} \quad \epsilon^{eq} \sim N(0, \tau \cdot \Sigma)$$
  - The precision of the equilibrium return estimates is assumed to be proportional to the variance of the returns.
  - τ is a scaling parameter

Black-Litterman Model: Views

- Investor views on a single asset affect many weights.
- "Portfolio views"
  - Investor views regarding the performance of K portfolios (e.g. each portfolio can contain only a single asset)
  - P: K x N matrix with portfolio weights
  - Q: K x 1 vector of views regarding the expected returns of these portfolios
- Investor views are assumed to be imprecise:

$$P \cdot \mu = Q + \epsilon^v \text{ where } \epsilon^v \sim N(0, \Omega)$$

  - Without loss of generality, Ω is assumed to be a diagonal matrix
  - ϵ^{eq} and ϵ^v are assumed to be independent

Black-Litterman Model: Posterior

- **Bayes rule:**

$$f(\theta|x) = \frac{f(\theta, x)}{f(x)} = \frac{f(x|\theta) \cdot f(\theta)}{f(x)}$$

- **Posterior distribution:**
  - If X_1, X_2 are normally distributed as:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$$

  - Then, the conditional distribution is given by

$$X_1|X_2 = x \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

Black-Litterman Model: Posterior

- The Black-Litterman formula for the posterior distribution of expected returns

$$
\begin{aligned}
&E[R|Q] \\
&= \ [(\tau \cdot \Sigma)^{-1} + P' \cdot \Omega^{-1} \cdot P]^{-1} \\
&\cdot [(\tau \cdot \Sigma)^{-1} \cdot \Pi + P' \cdot \Omega^{-1} \cdot Q]
\end{aligned}
$$

$$\mathrm{var}[R|Q] = [(\tau \cdot \Sigma)^{-1} + P' \cdot \Omega^{-1} \cdot P]^{-1}$$

Black Litterman: 2-asset Example

- Suppose you have a view on the equally weighted portfolio $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 = q + \varepsilon^v$
- Then

$$E[R|Q] = \left[(\tau \cdot \Sigma)^{-1} + \frac{1}{2\Omega}\right]^{-1} \cdot \left[(\tau \cdot \Sigma)^{-1} \cdot \Pi + \frac{q}{2\Omega}\right]$$

$$\mathrm{var}[R|Q] = \left[(\tau \cdot \Sigma)^{-1} + \frac{1}{2\Omega}\right]^{-1}$$

Advantages of Black-Litterman

- Returns are adjusted only partially toward the investor’s views using Bayesian updating
  - Recognizes that views may be due to estimation error
  - Only highly precise/confident views are weighted heavily.
- Returns are modified in way that is consistent with economic priors
  - Highly correlated sectors have returns modified in the same direction.
- Returns can be modified to reflect absolute or relative views.
- Resulting weight are reasonable and do not load up on estimation error.