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# LECTURE 3: ONE-PERIOD MODEL PRICING

# Overview: Pricing

1. LOOP, No arbitrage [L2,3]
2. Forwards [McD5]
3. Options: Parity relationship [McD6]
4. No arbitrage and existence of state prices [L2,3,5]
5. Market completeness and uniqueness of state prices
6. Unique  $q^*$
7. Four pricing formulas:  
state prices, SDF, EMM, beta pricing [L2,3,5,6]
8. Recovering state prices from options [DD10.6]

# Vector Notation

- Notation:  $y, x \in \mathbb{R}^n$ 
  - $y \geq x \Leftrightarrow y^i \geq x^i$  for each  $i = 1, \dots, n$
  - $y > x \Leftrightarrow y \geq x, y \neq x$
  - $y \gg x \Leftrightarrow y^i > x^i$  for each  $i = 1, \dots, n$
- Inner product
  - $y \cdot x = \sum yx$
- Matrix multiplication

# Three Forms of No-ARBITRAGE

## 1. Law of one Price (LOOP)

$$Xh = Xk \Rightarrow p \cdot h = p \cdot k$$

## 2. No strong arbitrage

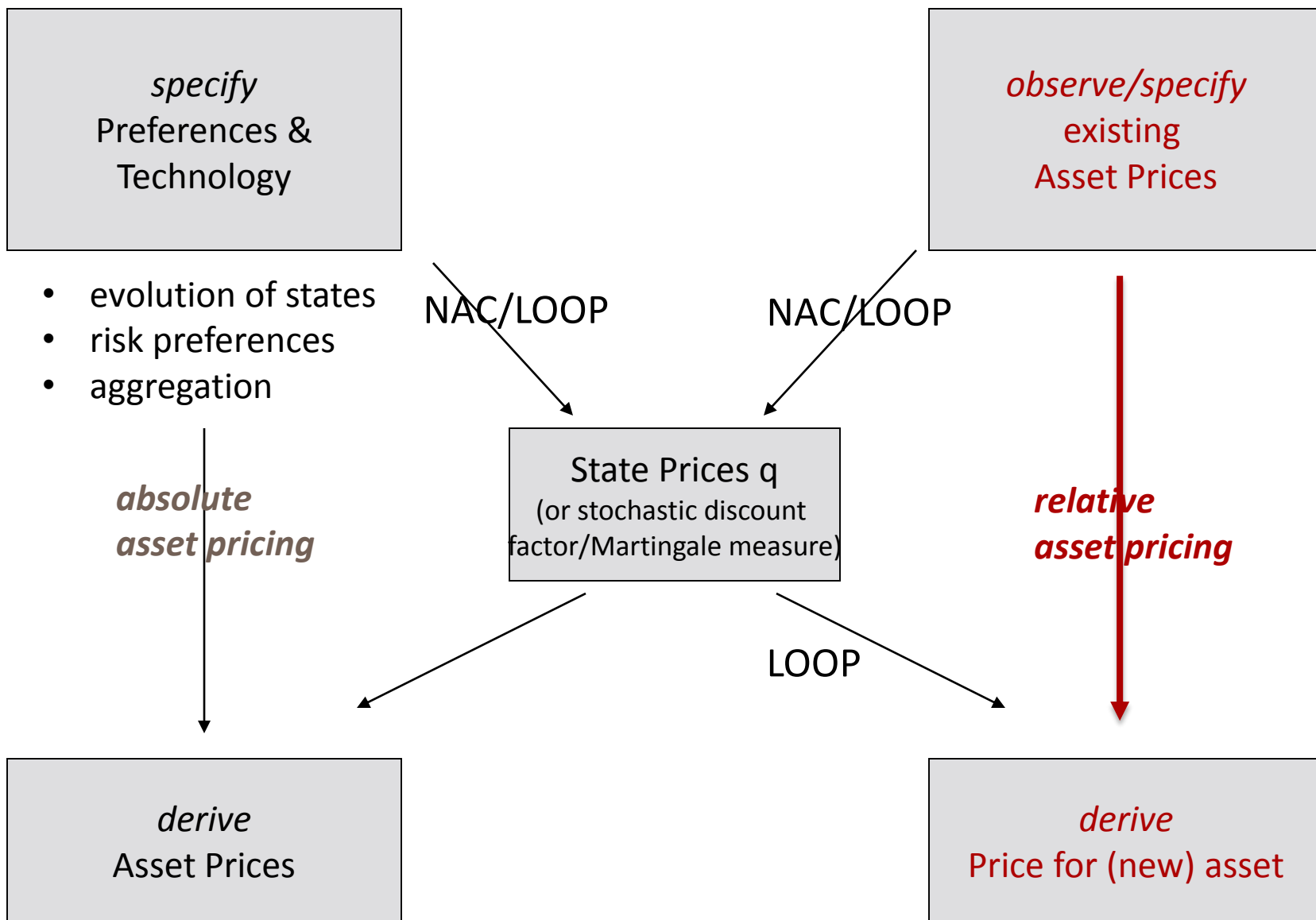
There exists no portfolio  $h$  which is a strong arbitrage, that is  $Xh \geq 0$  and  $p \cdot h < 0$

## 3. No arbitrage

There exists no strong arbitrage  
nor portfolio  $k$  with  $Xk > 0$  and  $p \cdot k \leq 0$

# Three Forms of No-ARBITRAGE

- Law of one price is equivalent to every portfolio with zero payoff has zero price.
- No arbitrage  $\Rightarrow$  no strong arbitrage  
No strong arbitrage  $\Rightarrow$  law of one price



Only works as long as market completeness doesn't change

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# Alternative ways to buy a stock

- Four different payment and receipt timing combinations:
  - Outright purchase: ordinary transaction
  - Fully leveraged purchase: investor borrows the full amount
  - Prepaid forward contract: pay today, receive the share later
  - Forward contract: agree on price now, pay/receive later
- Payments, receipts, and their timing:

**TABLE 5.1**

Four different ways to buy a share of stock that has price  $S_0$  at time 0. At time 0 you agree to a price, which is paid either today or at time  $T$ . The shares are received either at 0 or  $T$ . The interest rate is  $r$ .

<b>Description</b>	<b>Pay at Time:</b>	<b>Receive Security at Time:</b>	<b>Payment</b>
Outright Purchase	0	0	$S_0$ at time 0
Fully Leveraged Purchase	$T$	0	$S_0 e^{rT}$ at time $T$
Prepaid Forward Contract	0	$T$	?
Forward Contract	$T$	$T$	$? \times e^{rT}$



# Pricing prepaid forwards

- If we can price the *prepaid* forward ( $F^P$ ), then we can calculate the price for a forward contract:

$$F = \text{Future value of } F^P$$

- Pricing by analogy
  - In the absence of dividends, the timing of delivery is irrelevant
  - Price of the prepaid forward contract same as current stock price
  - $F_{0,T}^P = S_0$  (where the asset is bought at  $t = 0$ , delivered at  $t = T$ )

# Pricing prepaid forwards (cont.)

- Pricing by arbitrage
  - If at time  $t = 0$ , the prepaid forward price somehow exceeded the stock price, i.e.,  $F_{0,T}^P > S_0$ , an arbitrageur could do the following:

**TABLE 5.2** Cash flows and transactions to undertake arbitrage when the prepaid forward price,  $F_{0,T}^P$ , exceeds the stock price,  $S_0$ .

Transaction	Cash Flows	
	Time 0	Time $T$ (expiration)
Buy Stock @ $S_0$	$-S_0$	$+S_T$
Sell Prepaid Forward @ $F_{0,T}^P$	$+F_{0,T}^P$	$-S_T$
<b>Total</b>	$F_{0,T}^P - S_0$	0

# Pricing prepaid forwards (cont.)

- What if there are deterministic\* dividends? Is  $F_{0,T}^P = S_0$  still valid?
  - No, because the holder of the forward will not receive dividends that will be paid to the holder of the stock  $\Rightarrow F_{0,T}^P < S_0$ 

$$F_{0,T}^P = S_0 - PV(\text{all dividends paid from } t = 0 \text{ to } t = T)$$
  - For discrete dividends  $D_{t_i}$  at times  $t_i, i = 1, \dots, n$ 
    - The prepaid forward price:  $F_{0,T}^P = S_0 - \sum_{i=1}^n PV_{0,i}(D_{t_i})$   
(reinvest the dividend at risk-free rate)
  - For continuous dividends with an annualized yield  $\delta$ 
    - The prepaid forward price:  $F_{0,T}^P = S_0 e^{-\delta T}$   
(reinvest the dividend in this index. One has to invest only  $S_0 e^{-\delta T}$  initially)
  - Forward price is the future value of the *prepaid* forward:  $F_{0,T} = FV(F_{0,T}^P) = F_{0,T}^P \times e^{rT}$

NB: If dividends are stochastic, we cannot apply the one period model

# Creating a *synthetic* forward

- One can offset the risk of a forward by creating a *synthetic* forward to offset a position in the actual forward contract
- How can one do this? (assume continuous dividends at rate  $\delta$ )
  - Recall the long forward payoff at expiration  $S_T - F_{0,T}$
  - Borrow and purchase shares as follows:

**TABLE 5.3** Demonstration that borrowing  $S_0e^{-\delta T}$  to buy  $e^{-\delta T}$  shares of the index replicates the payoff to a forward contract,  $S_T - F_{0,T}$ .

Transaction	Cash Flows	
	Time 0	Time $T$ (expiration)
Buy $e^{-\delta T}$ Units of the Index	$-S_0e^{-\delta T}$	$+S_T$
Borrow $S_0e^{-\delta T}$	$+S_0e^{-\delta T}$	$-S_0e^{(r-\delta)T}$
<b>Total</b>	0	$S_T - S_0e^{(r-\delta)T}$

- Note that the total payoff at expiration is same as forward payoff
- This leads to: Forward = Stock – zero-coupon bond

# Other issues in forward pricing

- Does the forward price predict the future price?
  - According the formula  $F_{0,T} = S_0 e^{(r-\delta)T}$  the forward price conveys no additional information beyond what  $S_0, r, \delta$  provide
  - Moreover, if  $r < \delta$  the forward price underestimates the future stock price
  
- Forward pricing formula and cost of carry
  - Forward price =
 

$$\text{Spot price} + \underbrace{\text{Interest to carry the asset} - \text{asset lease rate}}_{\text{Cost of carry } (r - \delta)S}$$

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# Put-Call Parity

- For European options with the same strike price and time to expiration the parity relationship is:

$$\text{Call} - \text{Put} = \text{PV}(\text{Forward Px} - \text{Strike Px})$$

$$C(K, T) - P(K, T) = PV_{0,T}(F_{0,T} - K) = e^{-rT}(F_{0,T} - K)$$

ice ( $F_{0,T} = K$ ) creates a synthetic forward contract and hence must

- creates a synthetic forward contract and hence must have a zero price
- creates a synthetic forward contract and hence must have a zero price

# Parity for Options on Stocks

- If underlying asset is a stock and Div is the deterministic\* dividend stream, we can plug in  $e^{-rT} F_{0,T} = S_0 - PV_{0,T}(\text{Div})$  thus obtaining:

$$C(K, T) = P(K, T) + [S_0 - PV_{0,T}(\text{Div})] - e^{-rT} K$$

- For index options,  $S_0 - PV_{0,T}(\text{Div}) = S_0 e^{-\delta T}$ , therefore

$$C(K, T) = P(K, T) + S_0 e^{-\delta T} - e^{-rT} K$$

\* allows us to stay in one period setting



# Option price boundaries

- American vs. European

- Since an American option can be exercised at anytime, whereas a European option can only be exercised at expiration, an American option must always be at least as valuable as an otherwise identical European option:

$$\begin{aligned} C_A(S, K, T) &\geq C_E(S, K, T) \\ P_A(S, K, T) &\geq P_E(S, K, T) \end{aligned}$$

- Option price boundaries

- Call price cannot: be negative, exceed stock price, be less than price implied by put-call parity using zero for put price:

$$S > C_A(S, K, T) \geq C_E(S, K, T) > [PV_{0,T}(F_{0,T}) - PV_{0,T}(K)]^+$$

- Put price cannot: be negative, exceed strike price, be less than price implied by put-call parity using zero for call price:

$$K > P_A(S, K, T) \geq P_E(S, K, T) > [PV_{0,T}(K) - PV_{0,T}(F_{0,T})]^+$$

# Early exercise of American call

- Early exercise of American options
  - A non-dividend paying American call option should not be exercised early, because:
$$C_A \geq C_E = S_t - K + P_E + K(1 - e^{-r(T-t)}) > S_t - K$$
  - That means, one would lose money by exercising early instead of selling the option
- Caveats
  - If there are dividends, it may be optimal to exercise early
  - It may be optimal to exercise a non-dividend paying put option early if the underlying stock price is sufficiently low

# Options: Time to expiration

- Time to expiration
  - An American option (both put and call) with more time to expiration is at least as valuable as an American option with less time to expiration. This is because the longer option can easily be converted into the shorter option by exercising it early.
  - European call options on dividend-paying stock may be less valuable than an otherwise identical option with less time to expiration.

# Options: Time to expiration

- Time to expiration
  - When the strike price grows at the rate of interest, European call and put prices on a non-dividend paying stock increases with time.
    - Suppose to the contrary  $P(T) < P(t)$  for  $T > t$ , then arbitrage.
      - Buy  $P(T)$  and sell  $P(t)$  initially.
      - $S_t < K_t$ , keep stock and finance  $K_t$ , Time  $T$  value  $K_t e^{r(T-t)} = K_T$

0	t		T	
	$S_t < K_t$	$S_t > K_t$	$S_t < K_t$	$S_t > K_t$
$+P(t)$	$S_t - K_t$	0		
	$-S_t$		$+S_T$	
	$+K_t$		$-K_T$	
$-P(T)$			$\max\{K_T - S_T, 0\}$	
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$> 0$	0	0	$\geq 0$	$\geq 0$

# Options: Strike price

- Different strike prices ( $K_1 < K_2 < K_3$ ), for both European and American options

- A call with a low strike price is at least as valuable as an otherwise identical call with higher strike price:

$$C(K_1) \geq C(K_2)$$

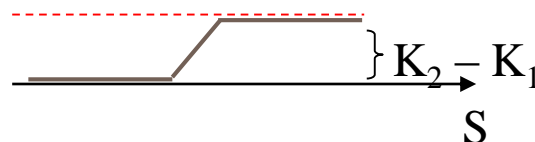
- A put with a high strike price is at least as valuable as an otherwise identical put with low strike price:

$$P(K_2) \geq P(K_1)$$

- The premium difference between otherwise identical calls with different strike prices cannot be greater than the difference in strike prices:

$$C(K_1) - C(K_2) \leq K_2 - K_1$$

- Price of a collar is not greater than its maximum payoff



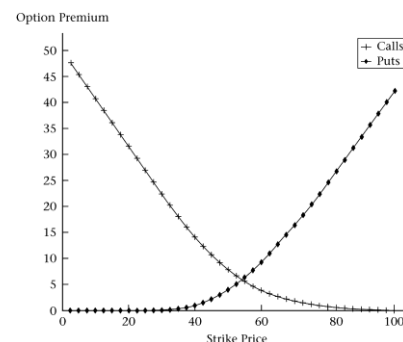
# Options: Strike price (cont.)

- Different strike prices ( $K_1 < K_2 < K_3$ ), for both European and American options
  - The premium difference between otherwise identical puts with different strike prices cannot be greater than the difference in strike prices:

$$P(K_2) - P(K_1) \leq K_2 - K_1$$

- Premiums decline at a decreasing rate for calls with progressively higher strike prices. (Convexity of option price with respect to strike price):

$$\frac{C(K_1) - C(K_2)}{K_1 - K_2} < \frac{C(K_2) - C(K_3)}{K_2 - K_3}$$



# Options: Strike price

- Proof: suppose to the contrary

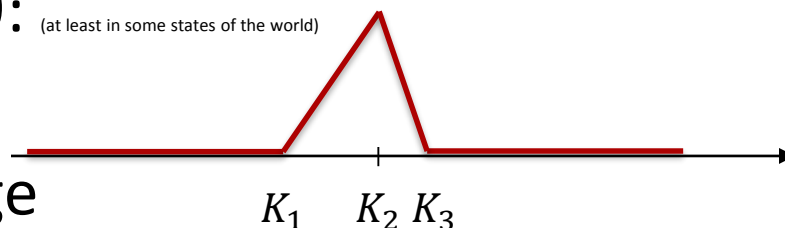
$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} \leq \frac{C(K_2) - C(K_3)}{K_3 - K_2}$$

- (Asymmetric) Butterfly spread

– Price  $\leq 0$ :

$$\frac{1}{K_2 - K_1} C(K_1) - \left( \frac{1}{K_2 - K_1} + \frac{1}{K_3 - K_2} \right) C(K_2) + \frac{1}{K_3 - K_2} C(K_3) \leq 0$$

– Payoff  $> 0$ : (at least in some states of the world)



–  $\Rightarrow$  arbitrage

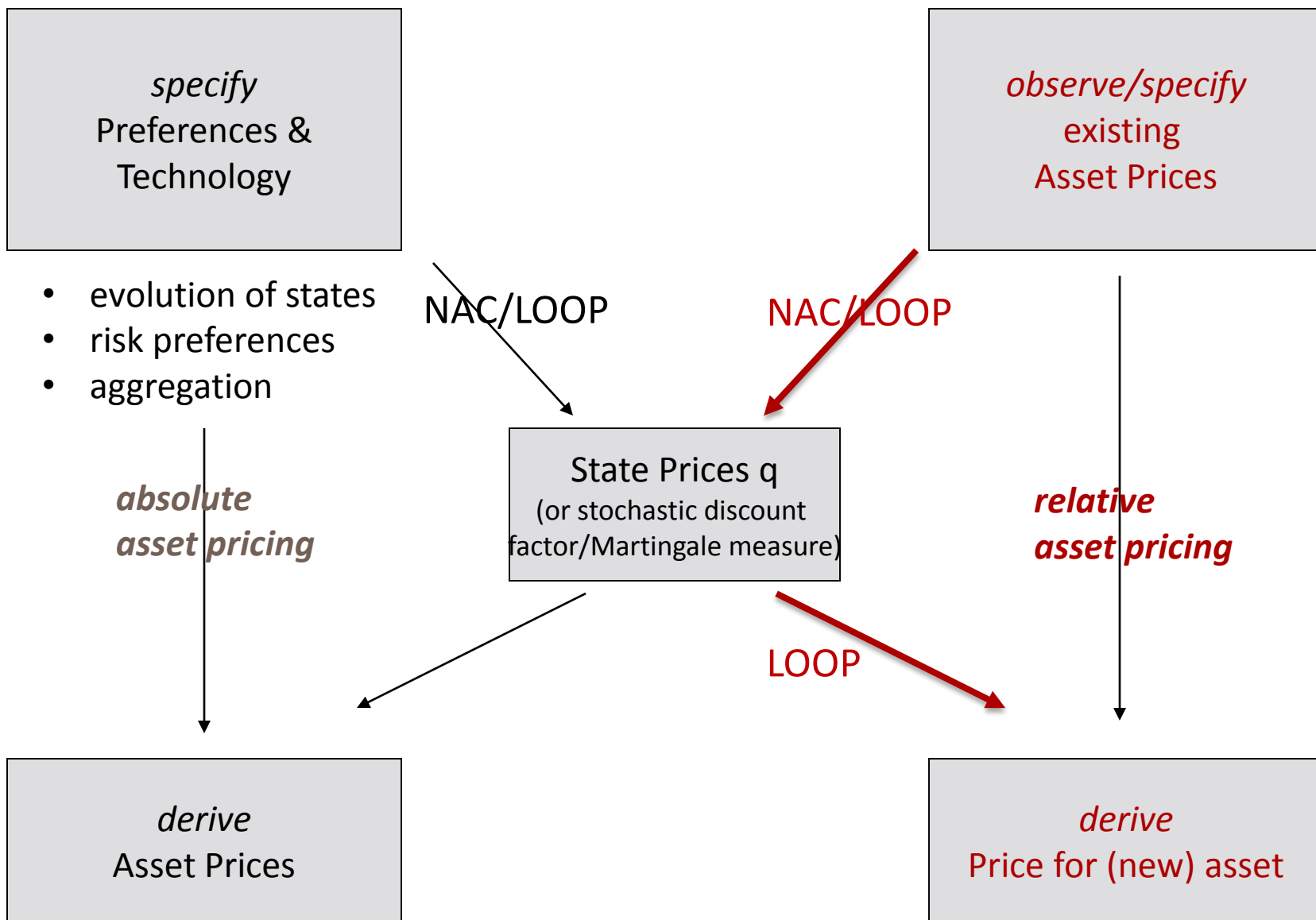
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## ... back to the big picture

- State space (evolution of states)
- (Risk) preferences
- Aggregation over different agents
- Security structure – prices of traded securities
- **Problem:**
  - Difficult to observe risk preferences
  - What can we say about existence of state prices without assuming specific utility functions/constraints for all agents in the economy



Only works as long as market completeness doesn't change

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## 2. No strong arbitrage

There exists no portfolio  $h$  which is a strong arbitrage, that is  $Xh \geq 0$  and  $p \cdot h < 0$

## 3. No arbitrage

There exists no strong arbitrage  
nor portfolio  $k$  with  $Xk > 0$  and  $p \cdot k \leq 0$

# Pricing

- Define for each  $z \in \langle X \rangle$ 
$$v(z) := \{p \cdot h : z = Xh\}$$
- If LOOP holds  $v(z)$  is a linear functional
  - Single-valued, because if  $h'$  and  $h''$  lead to same  $z$ , then price has to be the same
  - Linear on  $\langle X \rangle$
  - $v(0) = 0$
- Conversely, if  $v$  is a linear functional defined in  $\langle X \rangle$  then the law of one price holds.

# Pricing

- LOOP  $\Rightarrow v(Xh) = p \cdot h$
- A linear functional  $V \in \mathbb{R}^S$  is a valuation function if  $V(z) = v(z)$  for each  $z \in \langle X \rangle$
- $V(z) = q \cdot z$  for some  $q \in \mathbb{R}^S$ , where  $q^s = V(e_s)$ , and  $e_s$  is the vector with  $e_s^s = 1$  and  $e_s^i = 0$  if  $i \neq s$ 
  - $e_s$  is an Arrow-Debreu security
- $q$  is a vector of state prices
- $V$  extends  $v$  on  $\mathbb{R}^S$

# State prices $q$

- $q$  is a vector of state prices if  $p = X'q$ , that is  $p^j = x^j \cdot q$  for each  $j = 1, \dots, J$
- If  $V(z) = q \cdot z$  is a valuation functional then  $q$  is a vector of state prices
- Suppose  $q$  is a vector of state prices and LOOP holds. Then if  $z = Xh$  LOOP implies that

$$\begin{aligned} v(z) &= \sum_j h^j p^j \\ &= \sum_j \left( \sum_s x_s^j q_s \right) h^j = \sum_s \left( \sum_j x_s^j h^j \right) q_s = q \cdot z \end{aligned}$$

- $V(z) = q \cdot z$  is a valuation functional  $\Leftrightarrow q$  is a vector of state prices and LOOP holds

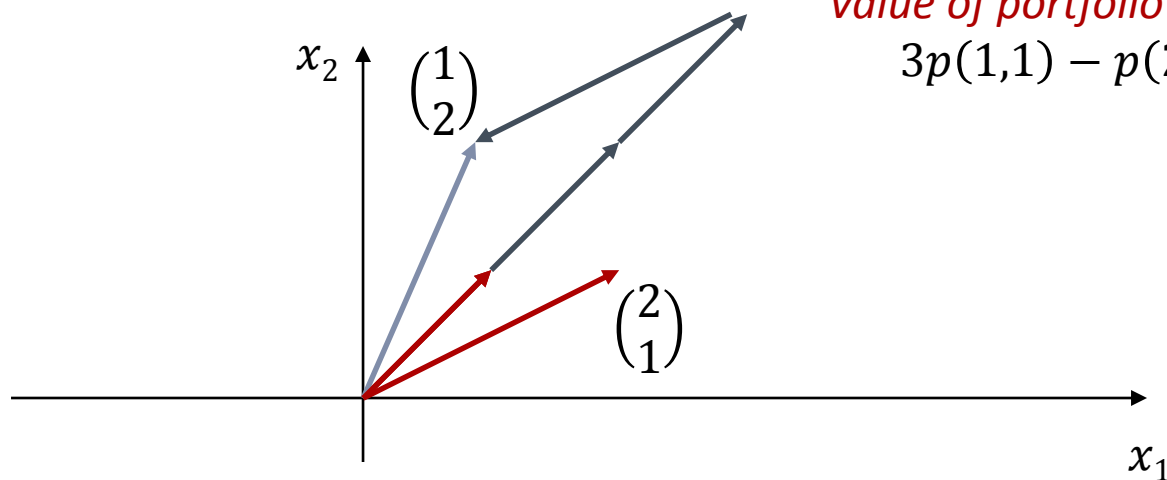
# State prices $q$

$$p(1,1) = q_1 + q_2$$

$$p(2,1) = 2q_1 + q_2$$

*Value of portfolio (1,2)*

$$3p(1,1) - p(2,1) = q_1 + 2q_2$$



# The Fundamental Theorem of Finance

- **Proposition 1.** Security prices exclude arbitrage if and only if there exists a valuation functional with  $q \gg 0$
- **Proposition 1'.** Let  $X$  be a  $S \times J$  matrix, and  $p \in \mathbb{R}^J$ . There is no  $h$  in  $\mathbb{R}^J$  satisfying  $h \cdot p \leq 0, Xh \geq 0$  and at least one strict inequality  $\Leftrightarrow$  there exists a vector  $q \in \mathbb{R}^S$  with  $q \gg 0$  and  $p = X'q$

No arbitrage  $\Leftrightarrow$  positive state prices

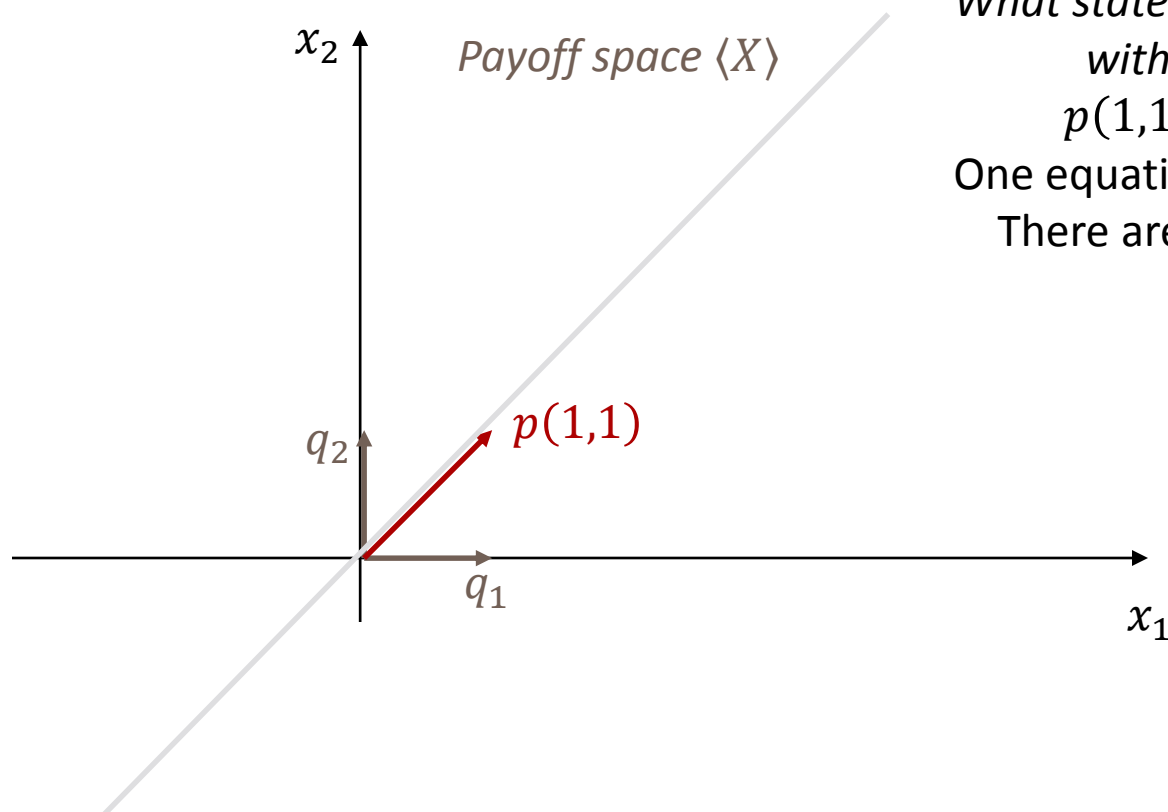


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# Multiple State Prices $q$ & Incomplete Markets

bond (1,1) only



What state prices are consistent  
with  $p(1,1)$ ?

$$p(1,1) = q_1 + q_2$$

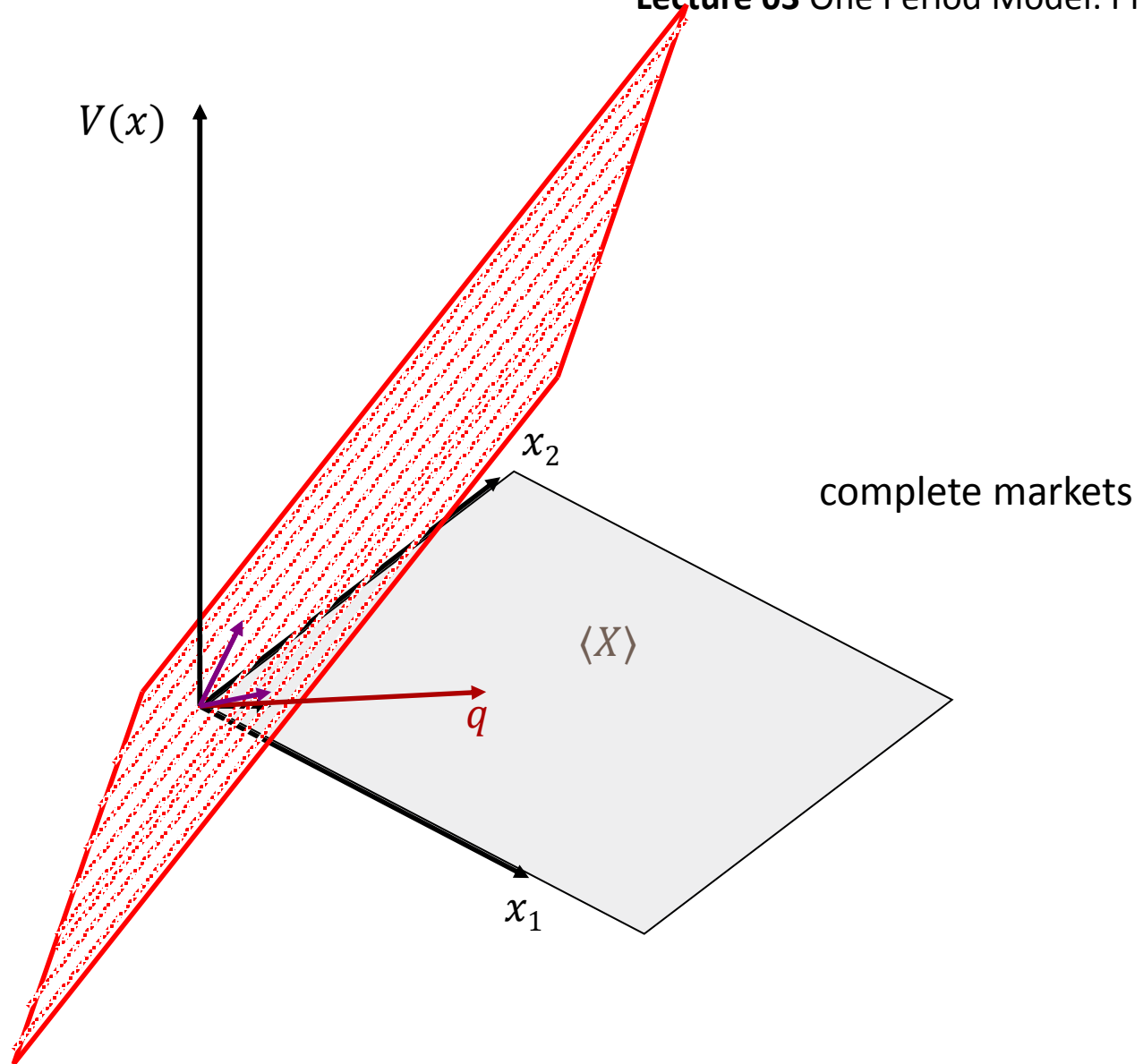
One equation – two unknowns  $q_1, q_2$

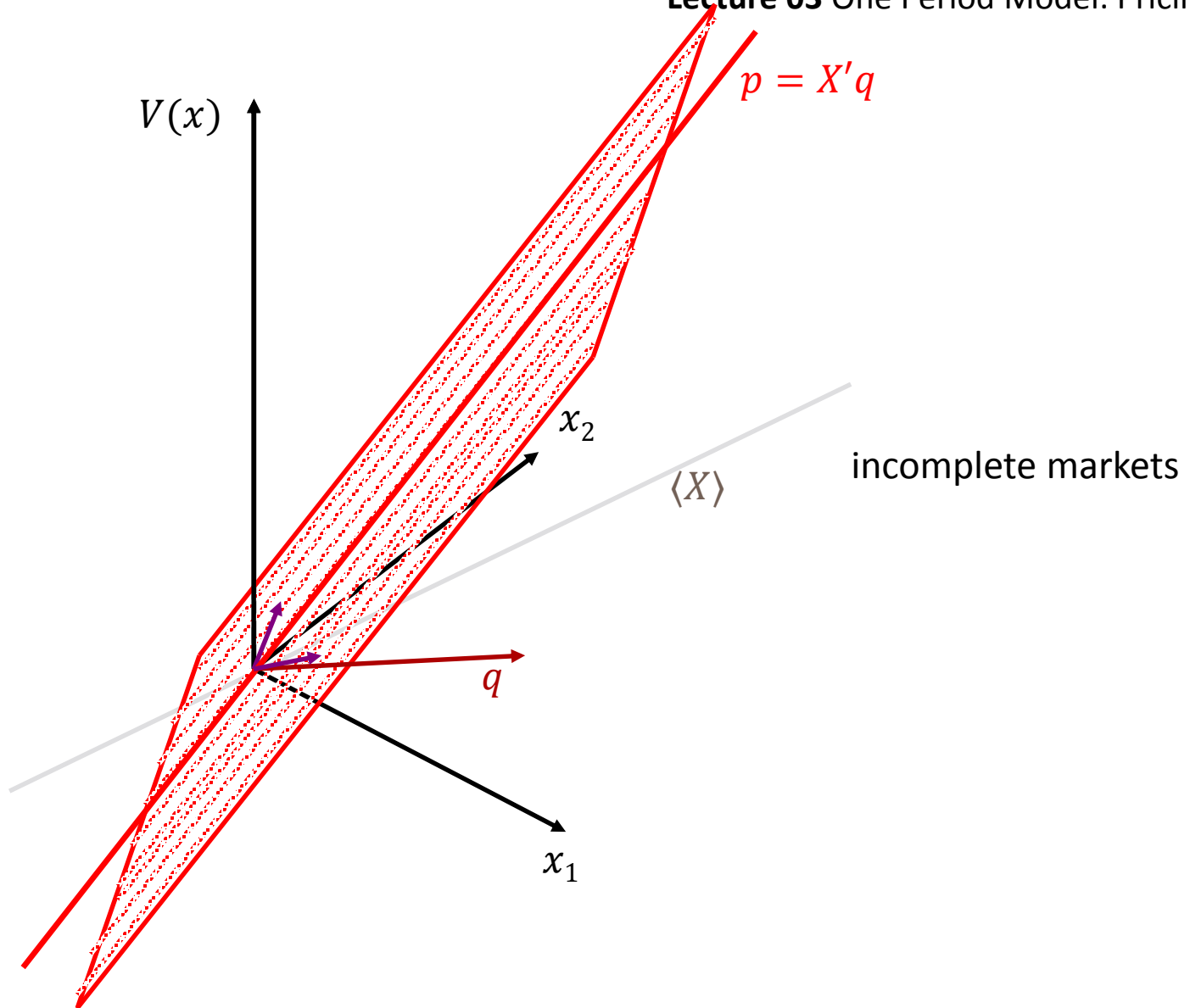
There are **(infinitely) many**.

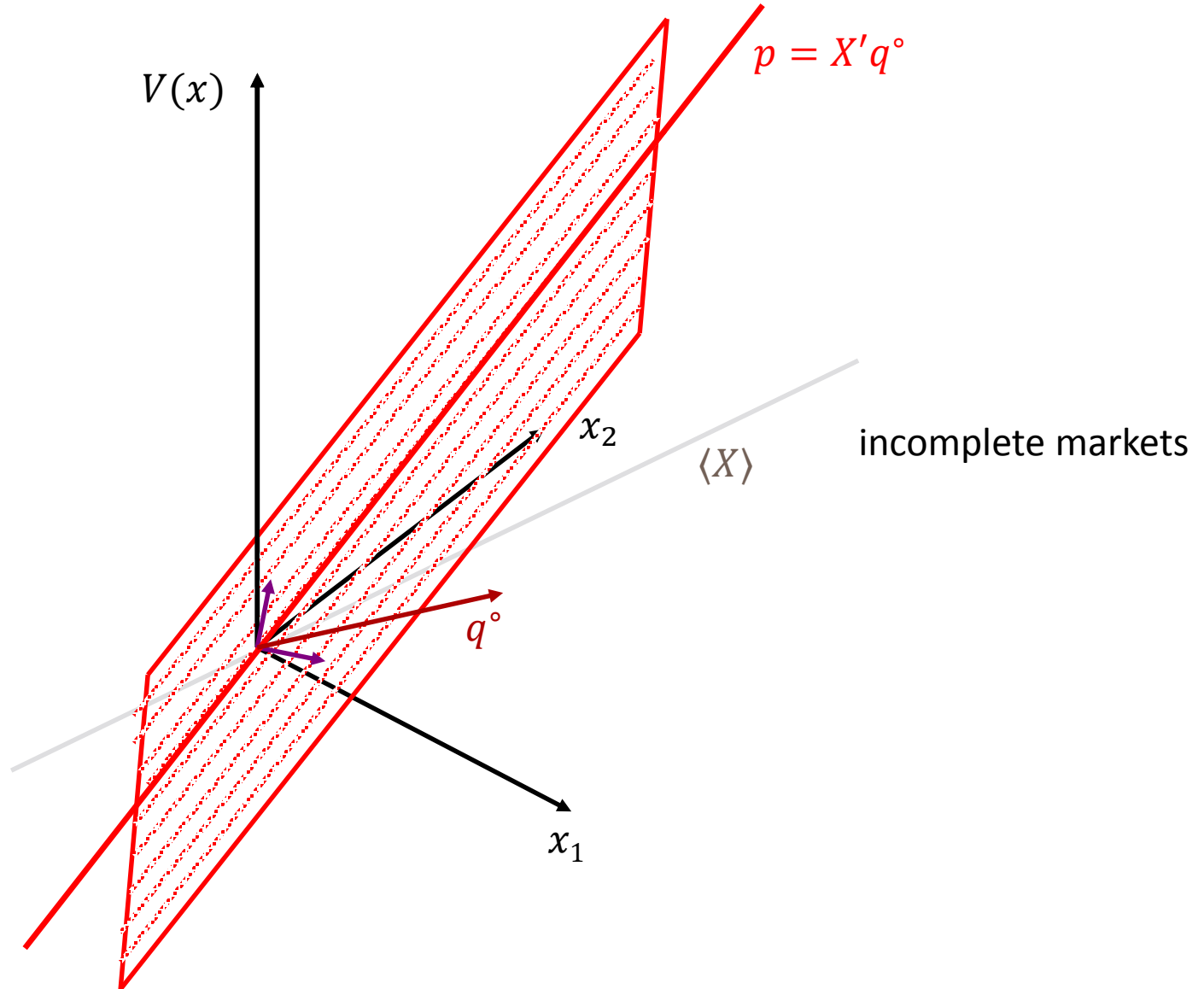
e.g. if  $p(1,1) = .9$

$$q_1 = .45, q_2 = .45,$$

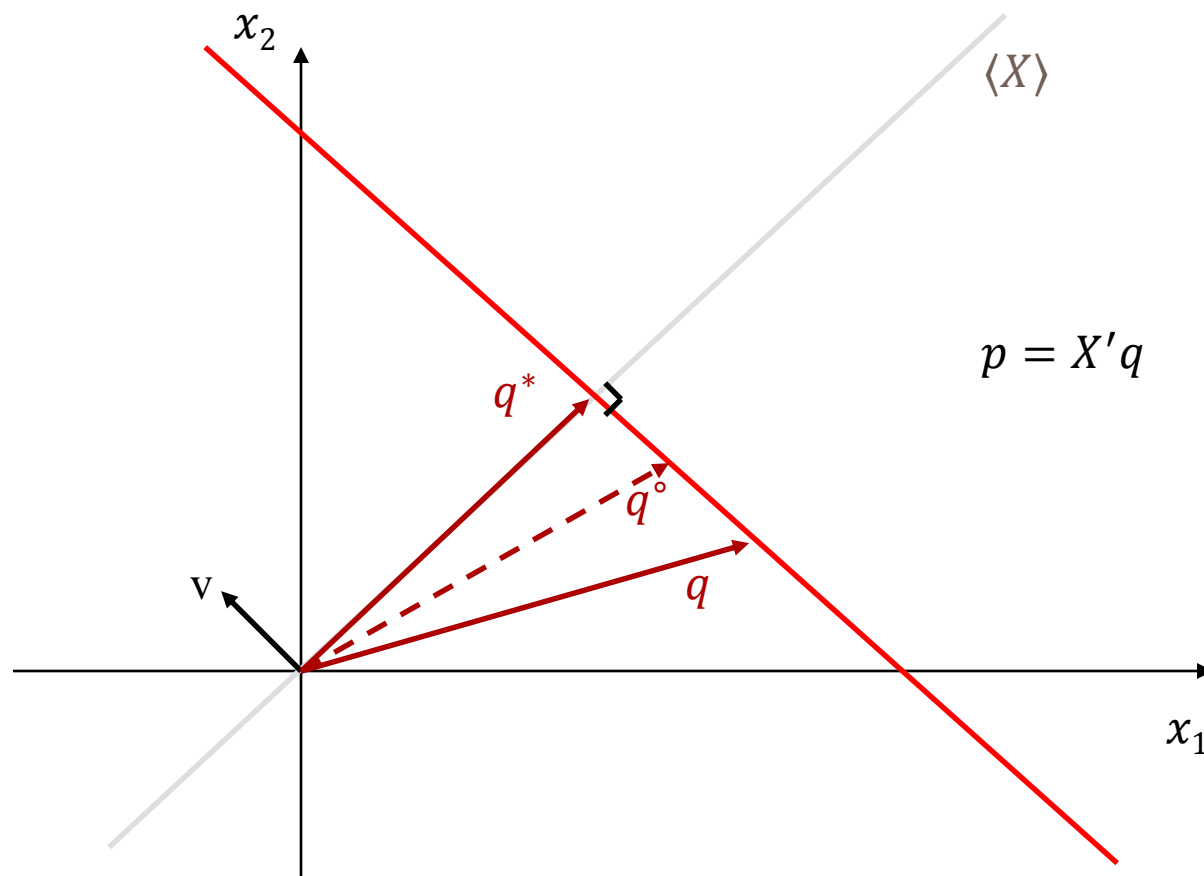
$$\text{or } q_1 = .35, q_2 = .55$$







# Multiple $q$ in incomplete markets



Many possible state price vectors s.t.  $p = X'q$ .  
 One is special:  $q^*$  - it can be replicated as a portfolio.

# Uniqueness and Completeness

- **Proposition 2.** If markets are complete, under no arbitrage there exists a *unique* valuation functional.
- If markets are not complete, then there exists  $v \in \mathbb{R}^S$  with  $0 = Xv$
- Suppose there is no arbitrage and let  $q \gg 0$  be a vector of state prices. Then  $q + \alpha v \gg 0$  provided  $\alpha$  is small enough, and  $p = X(q + \alpha v)$ . Hence, there are an infinite number of strictly positive state prices.

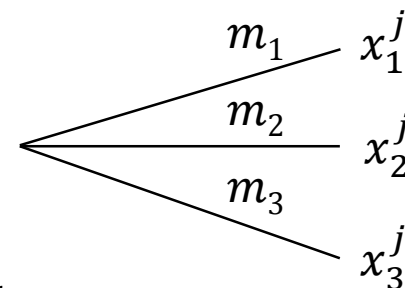
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# Four Asset Pricing Formulas

1. State prices  $p^j = \sum_S q_S x_S^j$
2. Stochastic discount factor  $p^j = E[mx^j]$



3. Martingale measure  $p^j = \frac{1}{1+r^f} E^{\hat{\pi}}[x^j]$

(reflect risk aversion by over(under)weighing the “bad(good)” states!)

4. State-price beta model  $E[R^j] - R^F = \beta^j E[R^* - R^f]$

(in returns  $R^j := \frac{x^j}{p^j}$ )

# 1. State Price Model

- ... so far price in terms of Arrow-Debreu (state) prices

$$p^j = \sum_s q_s x_s^j$$

## 2. Stochastic Discount Factor

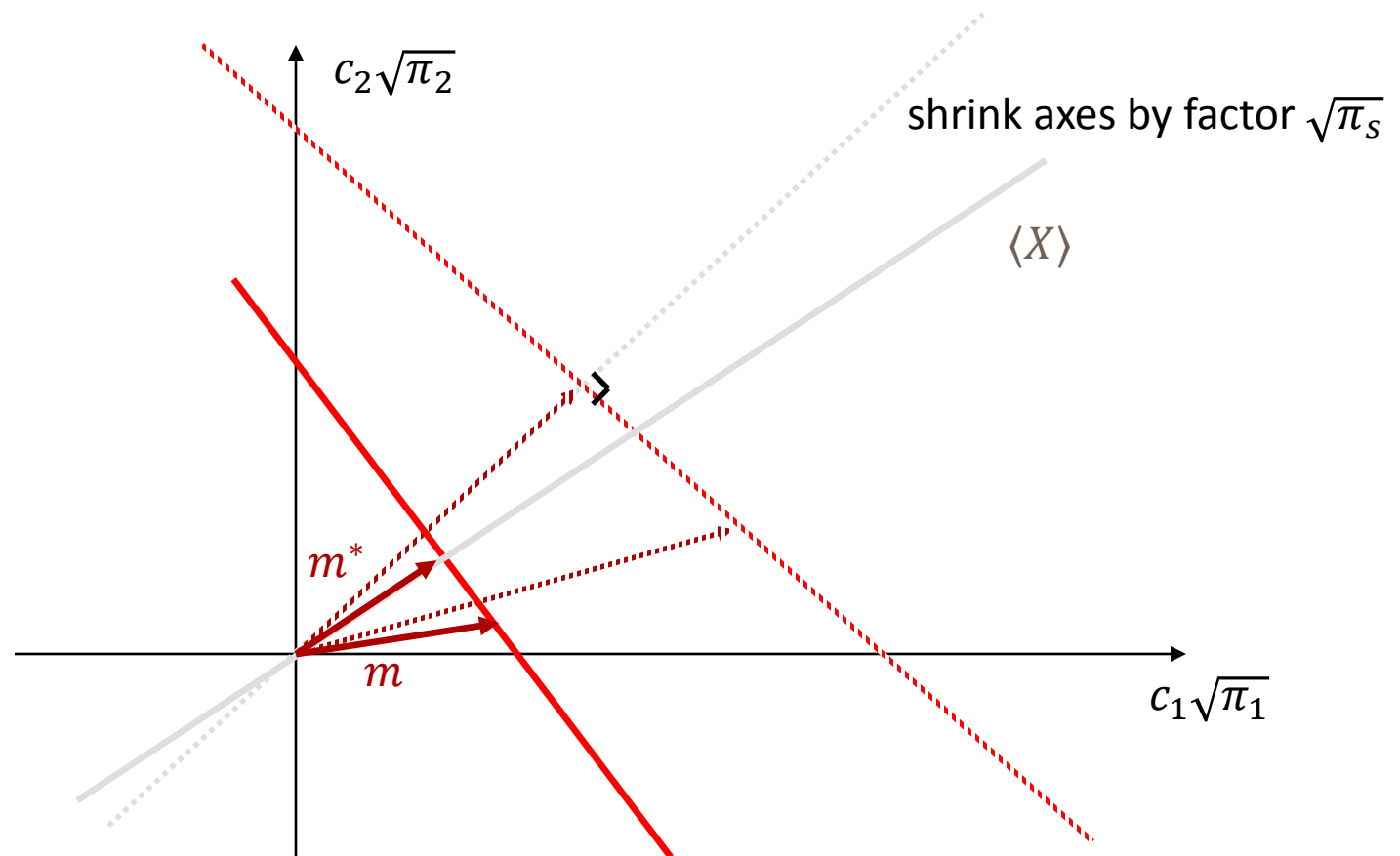
$$p^j = \sum_s q_s x_s^j = \sum_s \pi_s \frac{q_s}{\pi_s} x_s^j$$

- That is, stochastic discount factor  $m_s := \frac{q_s}{\pi_s}$

$$p^j = E[mx^j]$$

Now, probability inner product between  $m$  and  $x^j$

# 2. Stochastic Discount Factor



With  $m$ : Probability inner product = 0 (“probability orthogonal”)

# Risk-adjustment in payoffs

$$p = E[mx] = E[m]E[x] + \text{cov}[m, x]$$

Since  $p^{\text{bond}} = E[m \times 1]$ , the risk free rate  $\frac{1}{1+r^f} = \frac{1}{R^f} = E[m]$ .

$$p = \frac{E[x]}{R^f} + \text{cov}[m, x]$$

*Remarks:*

- (i) If risk-free rate does not exist,  $R^f$  is the shadow risk free rate
- (ii) Typically  $\text{cov}[m, x] < 0$ , which lowers price and increases return

# 3. Equivalent Martingale Measure

- Price of any asset  $p^j = \sum_s q_s x_s^j$
- Price of a bond  $p^{\text{bond}} = \sum_s q_s = \frac{1}{1+r^f}$

$$p^j = \frac{1}{1+r^f} \sum_s \frac{q_s}{\sum_{s'} q_{s'}} x_s^j = \frac{1}{1+r^f} E_{\hat{\pi}} [x^j]$$

$$\text{where } \hat{\pi}_s := \frac{q_s}{\sum_{s'} q_{s'}}$$

... in Returns:  $R^j = \frac{x^j}{p^j}$

$$E[mR^j] = 1, \quad R^f E[m] = 1 \Rightarrow E[m(R^j - R^f)] = 0$$

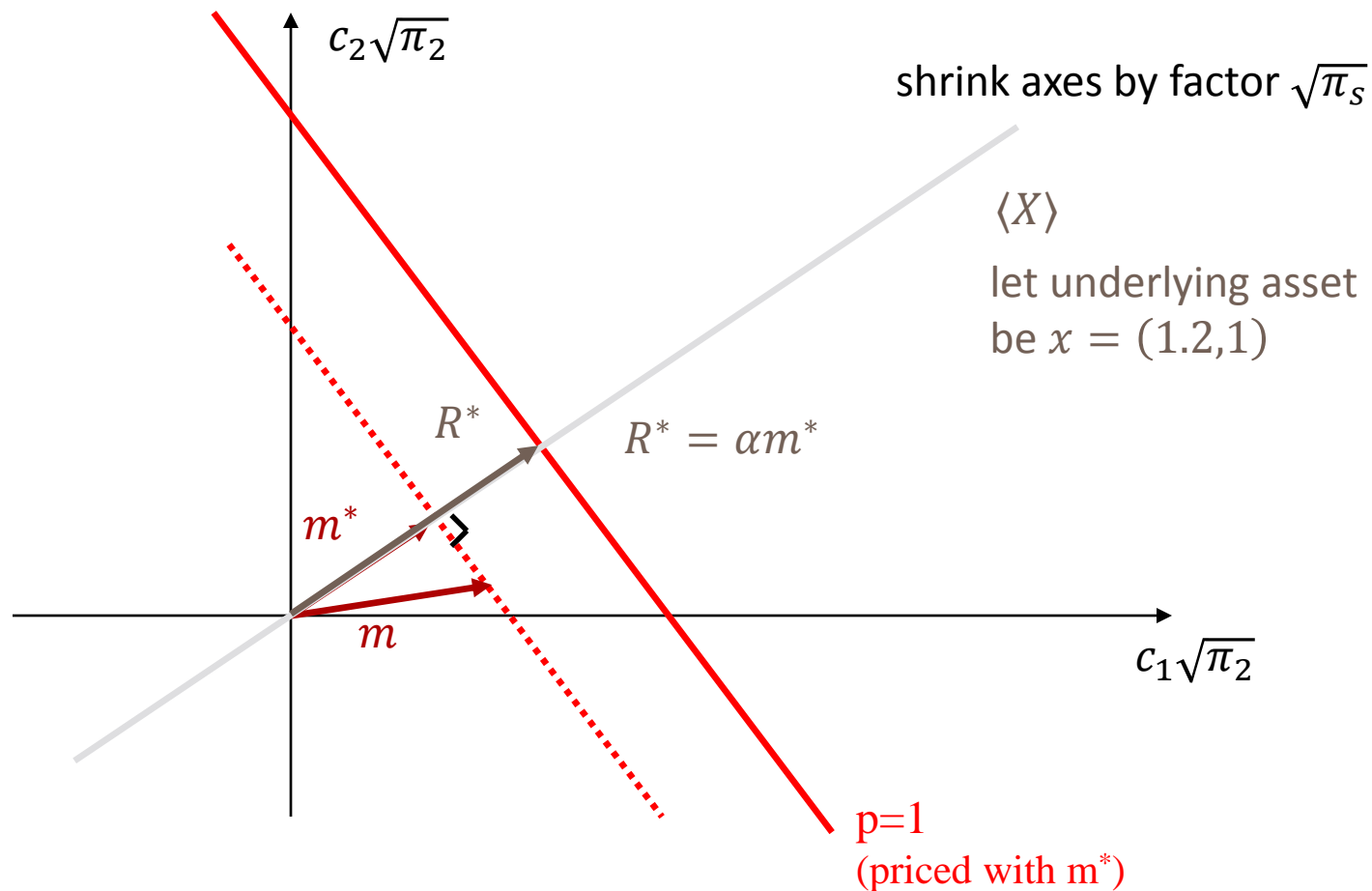
$$E[m](E[R^j] - R^f) + \text{cov}[m, R^j] = 0$$
$$\Rightarrow E[R^j] - R^f = -\frac{\text{cov}[m, R^j]}{E[m]}$$

(also holds for portfolios  $h$ )

*Note:*

- risk correction depends only on Cov of payoff/return with discount factor.
- Only compensated for taking on systematic risk not idiosyncratic risk.

# 4. State-price BETA Model



With  $m$ : Probability inner product = 0 (“probability orthogonal”)



## 4. State-price BETA Model

$$E[R^j] - R^f = -\frac{\text{cov}[m, R^j]}{E[m]}$$

(also holds for all portfolios  $h$ ,  
we can replace  $m$  with  $m^*$ )

Suppose (i)  $\text{var}[m^*] > 0$  and (ii)  $R^* = \alpha m^*$  with  $\alpha > 0$

$$E[R^h] - R^f = -\frac{\text{cov}[R^*, R^h]}{E[R^*]}$$

Define  $\beta^h := \frac{\text{cov}[R^*, R^h]}{\text{var}[R^*]}$  for any portfolio  $h$

# 4. State-price BETA Model

$$(2) \text{ for } R^h: E[R^h] - R^f = -\frac{\text{COV}[R^*, R^h]}{E[R^*]} = -\beta^h \frac{\text{var}[R^*]}{E[R^*]}$$

$$(2) \text{ for } R^*: E[R^*] - R^f = -\frac{\text{COV}[R^*, R^*]}{E[R^*]} = -\frac{\text{var}[R^*]}{E[R^*]}$$

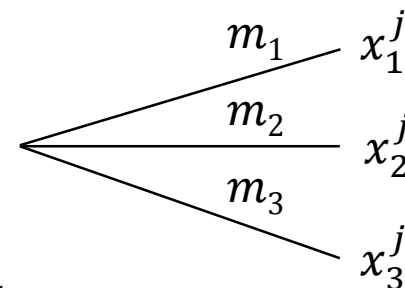
Hence,

$$E[R^h] - R^f = \beta^h E[R^* - R^f] \quad \text{where } \beta^h := \frac{\text{COV}[R^*, R^h]}{\text{var}[R^*]}$$

Regression  $R_s^h = \alpha^h + \beta^h (R^*)_s + \varepsilon_s$  with  $\text{cov}[R^*, \varepsilon] = E[\varepsilon] = 0$   
 very general – but what is  $R^*$  in reality?

# Four Asset Pricing Formulas

1. State prices  $p^j = \sum_S q_S x_S^j$
2. Stochastic discount factor  $p^j = E[mx^j]$



3. Martingale measure  $p^j = \frac{1}{1+r^f} E^{\hat{\pi}} [x^j]$

(reflect risk aversion by over(under)weighing the “bad(good)” states!)

4. State-price beta model  $E[R^j] - R^F = \beta^j E[R^* - R^f]$

(in returns  $R^j := \frac{x^j}{p^j}$ )

# What do we know about $q, m, \hat{\pi}, R^*$ ?

- Main results so far
  - Existence  $\Leftrightarrow$  no arbitrage
    - Hence, single factor only
    - But doesn't famous Fama-French factor model have 3 factors?
    - Additional factors are due to time-variation (wait for multi-period model)
  - Uniqueness if markets are complete

# Different Asset Pricing Models

$$p_t = E[m_{t+1}x_{t+1}] \Rightarrow E[R^h] - R^f = \beta^h E[R^* - R^f]$$

$$\text{where } m_{t+1} = f(\dots) \text{ and } \beta^h = \frac{\text{COV}[R^*, R^h]}{\text{var}[R^*]}$$

**$f(\dots)$  = asset pricing model**

**General Equilibrium**

$$f(\dots) = \frac{\text{MRS}}{\pi}$$

**Factor Pricing Model**

$$a + b_1 f_{1,t+1} + b_2 f_{2,t+1}$$

**CAPM**

$$a + b_1 f_{1,t+1} = a + b_1 R^M$$

**CAPM**

$$R^* = R^f \frac{a + b_1 R^M}{a + b_1 R^f}$$

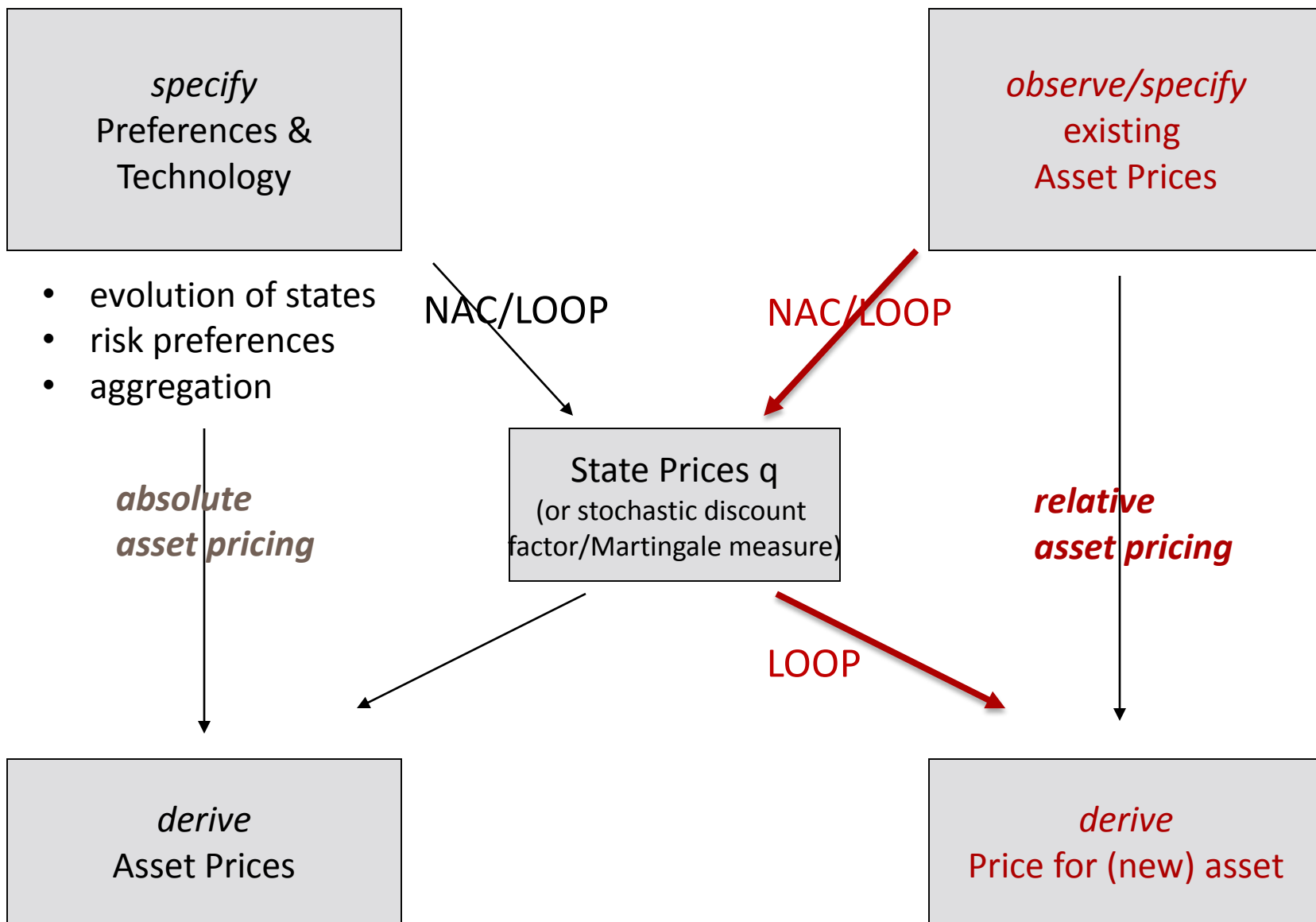
where  $R^m$  is market return  
 is  $b_1 \geq 0$ ?

# Different Asset Pricing Models

- Theory
  - All economics and modeling is determined by
$$m_{t+1} = a + \mathbf{b}' \mathbf{f}$$
  - Entire content of model lies in restriction of SDF
- Empirics
  - $m^*$  (which is a portfolio payoff) prices as well as  $m$  (which is e.g. a function of income, investment etc.)
  - measurement error of  $m^*$  is smaller than for any  $m$
  - Run regression on *returns* (portfolio payoffs)!  
(e.g. Fama-French three factor model)

# Overview: Pricing - one period model

1. LOOP, No arbitrage
2. Forwards
3. Options: Parity relationship
4. No arbitrage and existence of state prices
5. Market completeness and uniqueness of state prices
6. Unique  $q^*$
7. Four pricing formulas:  
state prices, SDF, EMM, beta pricing
8. Recovering state prices from options



Only works as long as market completeness doesn't change

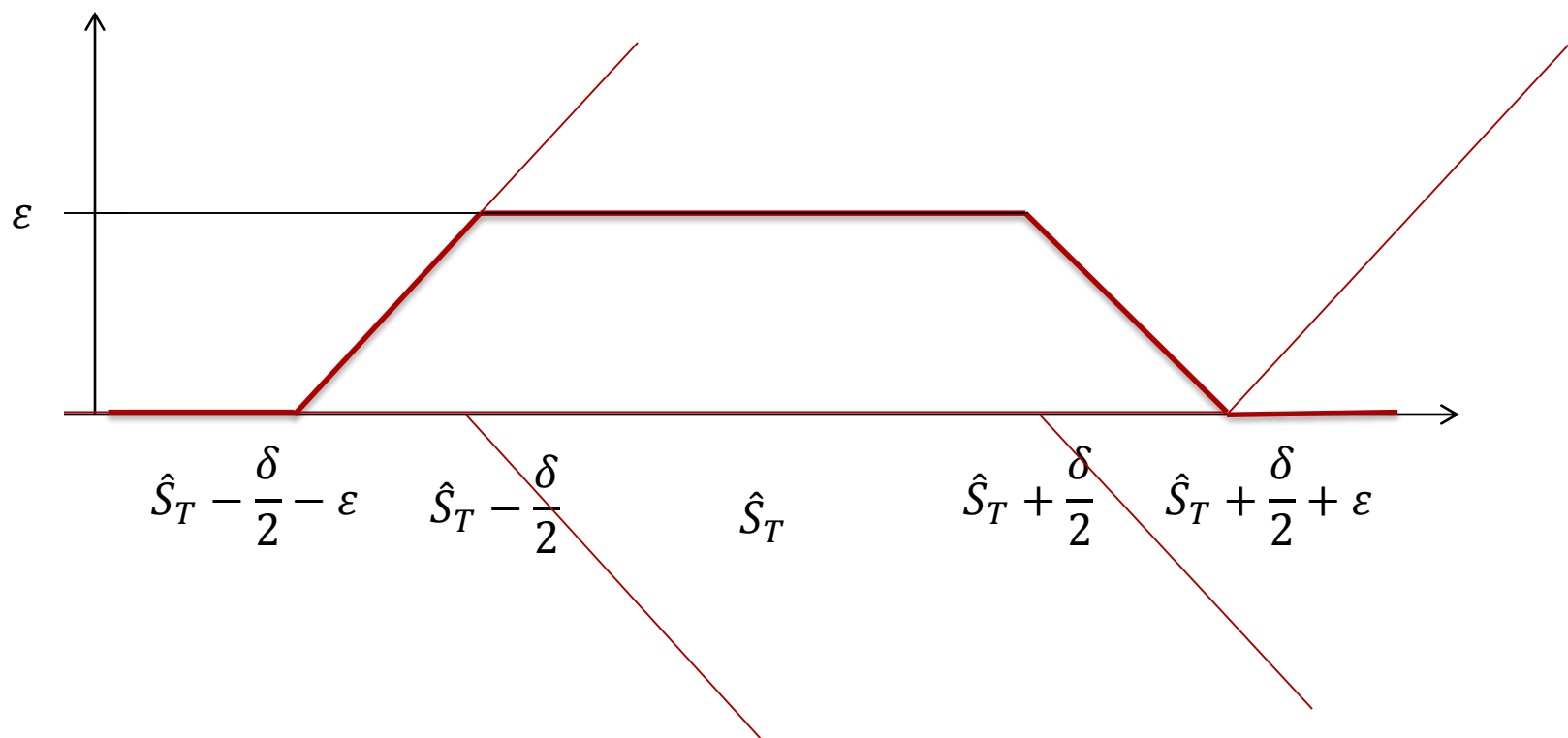


# Recovering State Prices from Option Prices

- Suppose that  $S_T$ , the price of the underlying portfolio (we may think of it as a proxy for price of “market portfolio”), assumes a “continuum” of possible values.
- Suppose there are a “continuum” of call options with different strike/exercise prices  $\Rightarrow$  markets are complete
- Let us construct the following portfolio:  
 for some small positive number  $\varepsilon > 0$ 
  - Buy one call with  $K = \hat{S}_T - \frac{\delta}{2} - \varepsilon$
  - Sell one call with  $K = \hat{S}_T - \frac{\delta}{2}$
  - Sell one call with  $K = \hat{S}_T + \frac{\delta}{2}$
  - Buy one call with  $K = \hat{S}_T + \frac{\delta}{2} + \varepsilon$

# Recovering State Prices ... (ctd)

Payoff of the portfolio



# Recovering State Prices ... (ctd)

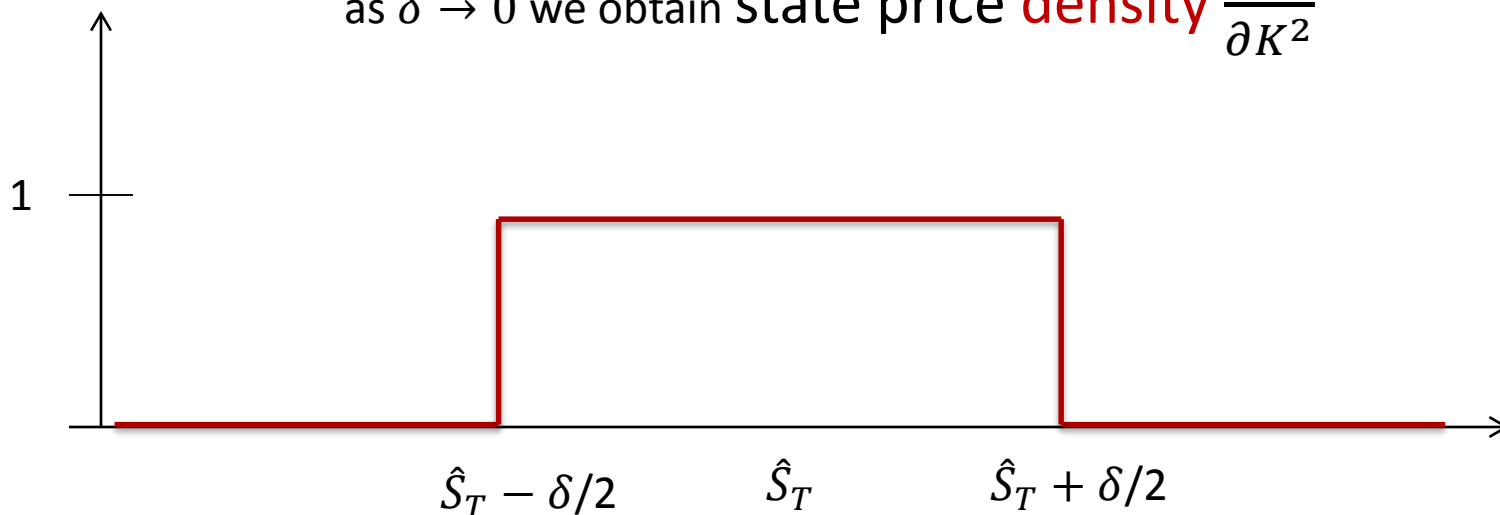
- Let us thus consider buying  $\frac{1}{\varepsilon}$  units of the portfolio.
  - The total payment, when  $\hat{S}_T - \frac{\delta}{2} \leq S_T \leq \hat{S}_T + \frac{\delta}{2}$  is  $\varepsilon \cdot \frac{1}{\varepsilon} = 1$ , for any  $\varepsilon$
  - Letting  $\varepsilon \rightarrow 0$  eliminates payments in the regions  $S_T \in \left[ \hat{S}_T - \frac{\delta}{2} - \varepsilon, \hat{S}_T - \frac{\delta}{2} \right)$  and  $S_T \in \left( \hat{S}_T + \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} + \varepsilon \right]$
- The value of  $\frac{1}{\varepsilon}$  units of this portfolio is
 
$$\frac{1}{\varepsilon} \left\{ C \left( S, K = \hat{S}_T - \frac{\delta}{2} - \varepsilon \right) - C \left( S, K = \hat{S}_T - \frac{\delta}{2} \right) \right\}$$

# Recovering State Prices ... (ctd)

- Taking the limit  $\varepsilon \rightarrow 0$

$$\begin{aligned}
 &= -\lim_{\varepsilon \rightarrow 0} \frac{C\left(S, K = \hat{S}_T - \frac{\delta}{2}\right) - C\left(S, K = \hat{S}_T - \frac{\delta}{2} - \varepsilon\right)}{\varepsilon} + \lim_{\varepsilon \rightarrow 0} \frac{C\left(S, K = \hat{S}_T + \frac{\delta}{2} + \varepsilon\right) - C\left(S, K = \hat{S}_T + \frac{\delta}{2}\right)}{\varepsilon} \\
 &= -\frac{\partial C\left(S, K = \hat{S}_T - \frac{\delta}{2}\right)}{\partial K} + \frac{\partial C\left(S, K = \hat{S}_T + \frac{\delta}{2}\right)}{\partial K}
 \end{aligned}$$

as  $\delta \rightarrow 0$  we obtain state price density  $\frac{\partial^2 C}{\partial K^2}$



# Recovering State Prices ... (ctd.)

- Evaluate the following cash flow

$$CF_T = \begin{cases} 0 & S_T \notin \left[ \hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} \right] \\ 50000 & S_T \in \left[ \hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} \right] \end{cases}$$

- Value of this cash flow today

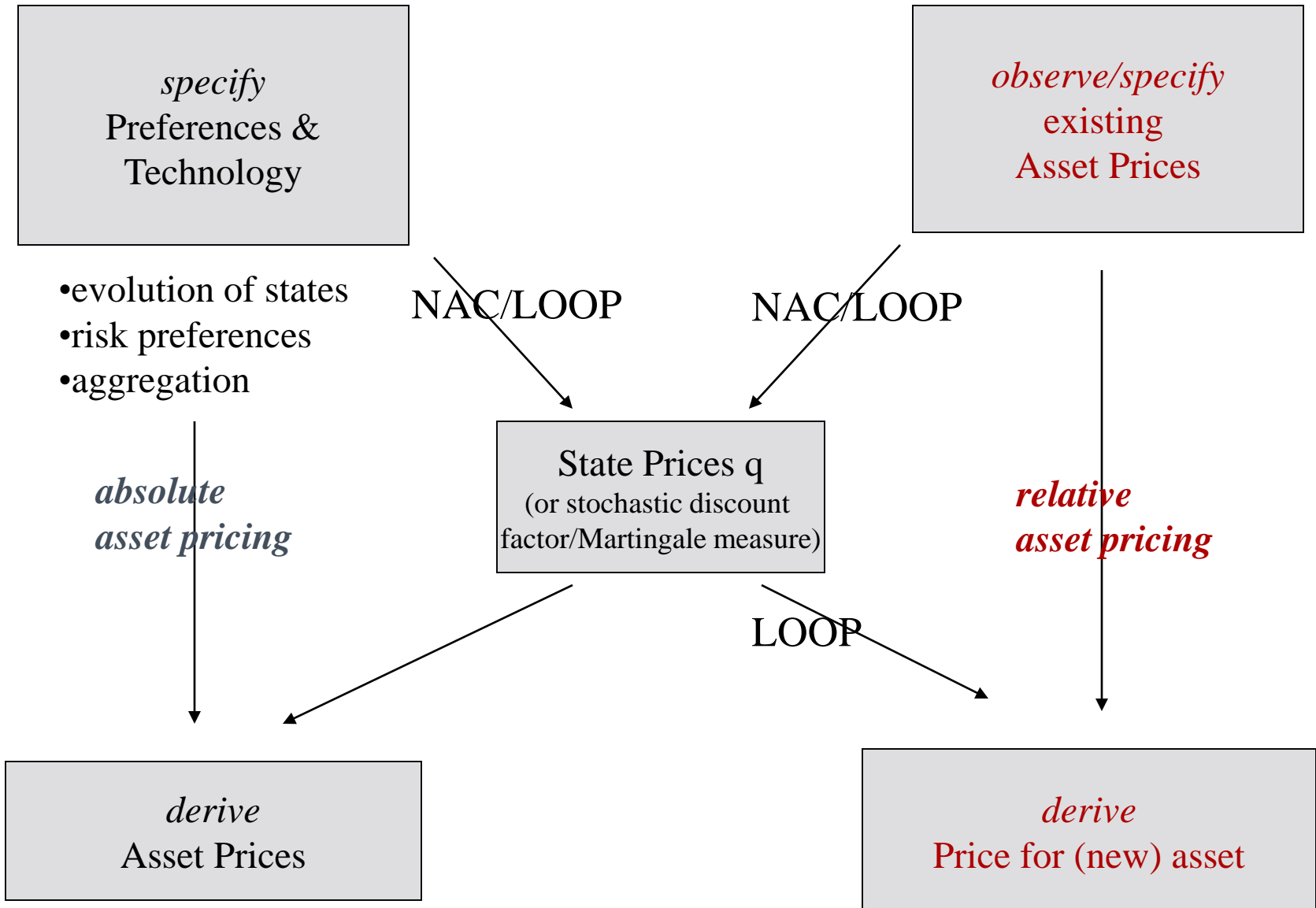
$$50000 \left[ \frac{\partial C}{\partial K} \left( S, K = \hat{S}_T + \frac{\delta}{2} \right) - \frac{\partial C}{\partial K} \left( S, K = \hat{S}_T - \frac{\delta}{2} \right) \right]$$

$$q(S_T^1, S_T^2) = \frac{\partial C}{\partial K} (S, K = S_T^1) - \frac{\partial C}{\partial K} (S, K = S_T^2)$$

**Table 8.1** Pricing an Arrow-Debreu State Claim

E	C(S,E)	Cost of position	Payoff if $S_T =$							$\Delta C$	$\Delta (\Delta C) = q_s$
			7	8	9	10	11	12	13		
7	3.354									-0.895	
8	2.459									-0.789	0.106
9	1.670	+1.670	0	0	0	1	2	3	4	-0.625	0.164
10	1.045	-2.090	0	0	0	0	-2	-4	-6	-0.441	<b>0.184</b>
11	0.604	+0.604	0	0	0	0	0	1	2	-0.279	0.162
12	0.325									-0.161	0.118
13	0.164	<b>0.184</b>	0	0	0	1	0	0	0		

 Note  $\Delta K = 1$



Only works as long as market completeness doesn't change