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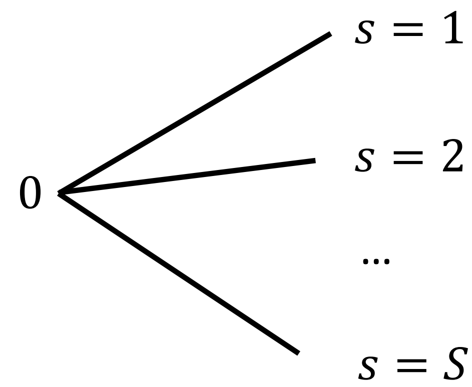
LECTURE 2: ONE PERIOD MODEL STRUCTURE

Overview

1. Securities Structure
   - Arrow-Debreu securities structure
   - Redundant securities
   - Market completeness
   - Completing markets with options

The Economy



- State space (Evolution of states)
  - Two dates: $t = 0,1$
  - S states of the world at time $t = 1$
- Preferences
  - $U(c_0, c_1, \dots, c_S)$
  - $MRS_{s,0}^A = -\frac{\partial U^A / \partial c_s^A}{\partial U^A / \partial c_0^A}$ (slope of indifference curve)
- Security structure
  - Arrow-Debreu economy
  - General security structure

Security Structure

- Security *j* is represented by a payoff *vector*

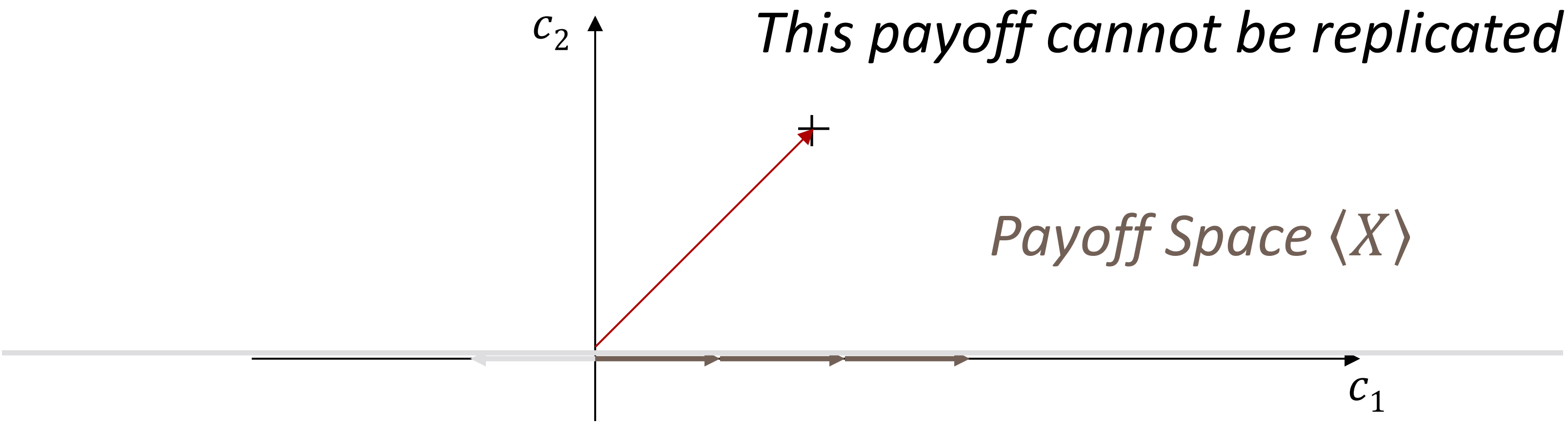
$$\left(x_1^j, x_2^j, \dots, x_S^j\right)'$$

- Security structure is represented by payoff matrix

$$X = \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^{J-1} & x_1^J \\ x_2^1 & & & & x_2^J \\ \vdots & & \ddots & & \vdots \\ x_{S-1}^1 & & & & x_{S-1}^J \\ x_S^1 & x_S^2 & \cdots & x_S^{J-1} \cdots & x_S^J \end{pmatrix}$$

Arrow-Debreu Security Structure in $\mathbb{R}^2$

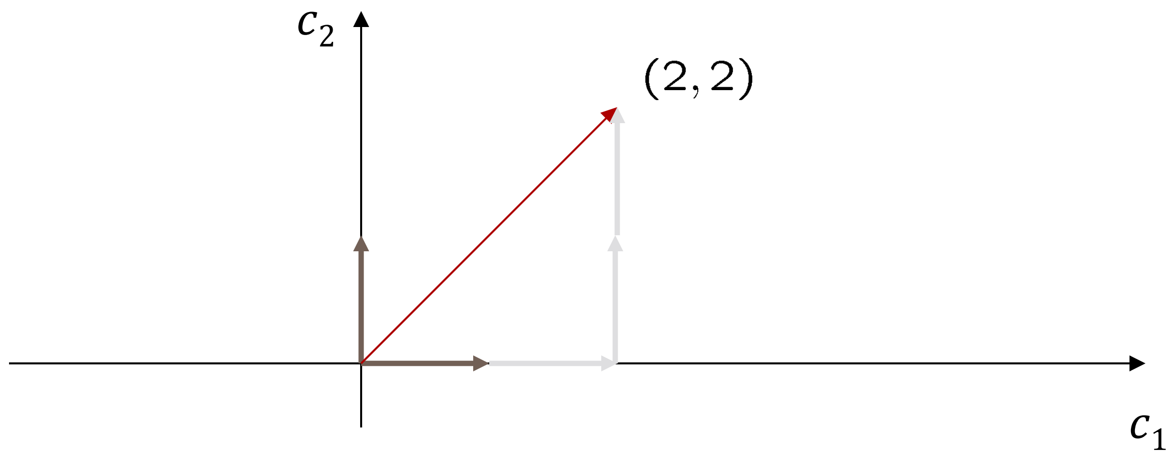
One A-D asset $e_1 = (1,0)'$



Markets are **incomplete**

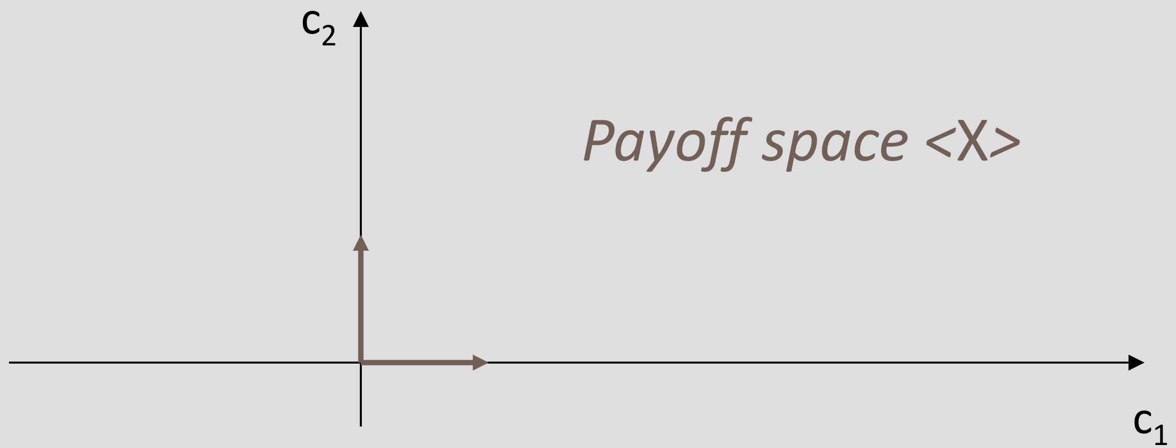
Arrow-Debreu Security Structure in $\mathbb{R}^2$

Add second A-D asset $e_2 = (0,1)'$ to $e_1 = (1,0)'$



Arrow-Debreu Security Structure in $\mathbb{R}^2$

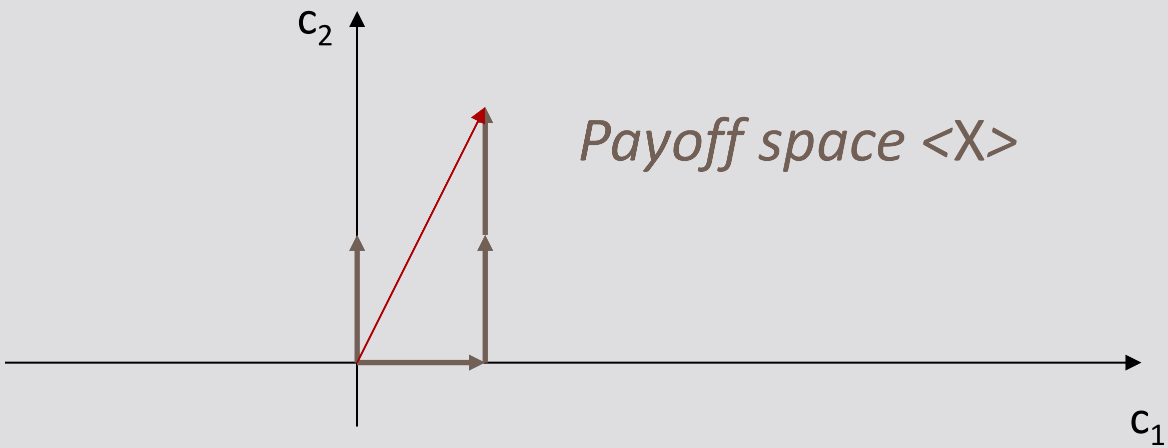
Add second A-D asset $e_2 = (0,1)'$ to $e_1 = (1,0)'$



Any payoff can be replicated with two A-D securities

Arrow-Debreu Security Structure in $\mathbb{R}^2$

Add second asset $(1,2)'$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



*New asset is **redundant** – it does not enlarge the payoff space*

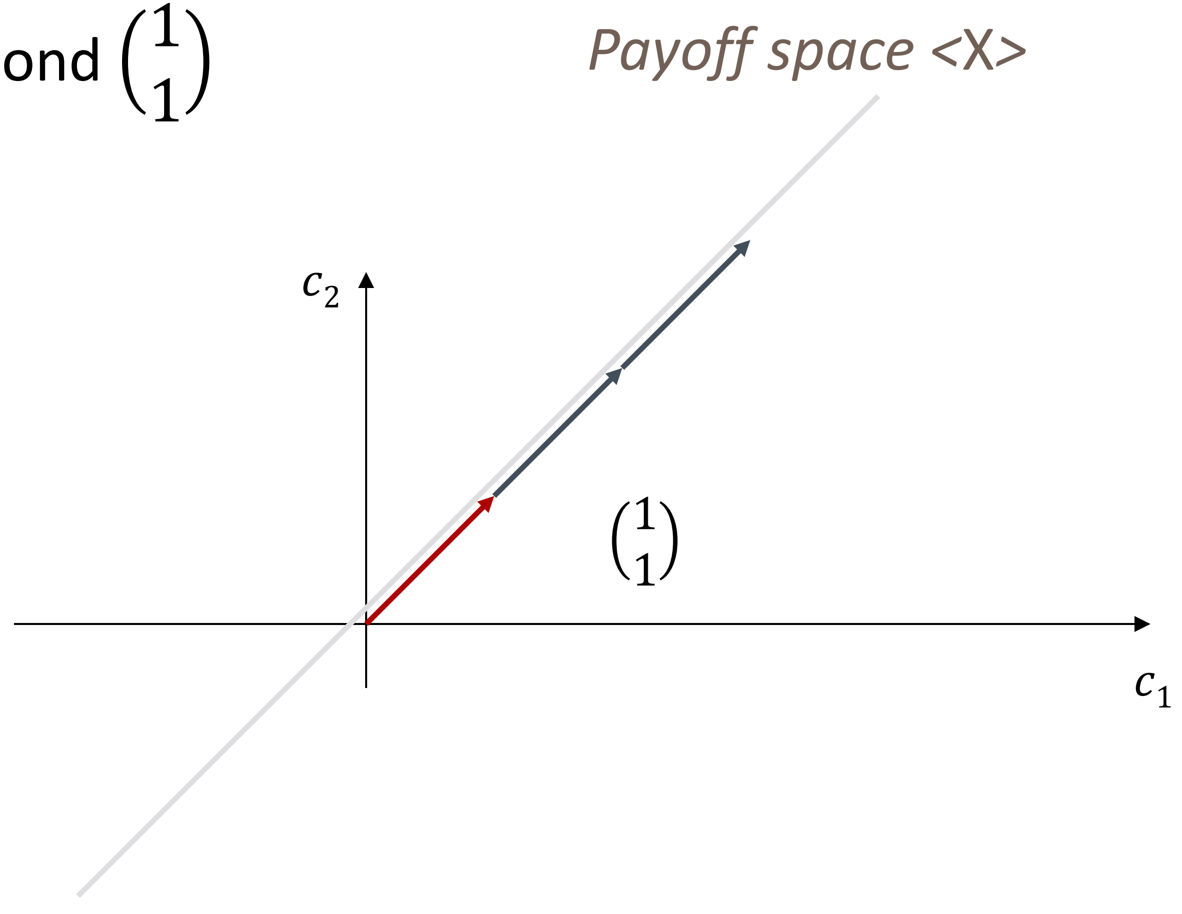
Arrow-Debreu Security Structure

$$X = \begin{pmatrix} 1 & 0 & \ddots \\ 0 & \ddots & 0 \\ \ddots & 0 & 1 \end{pmatrix}$$

- S Arrow-Debreu securities
- each state s can be insured individually
- All payoffs are linearly independent
- Rank of $X = S$
- Markets are complete

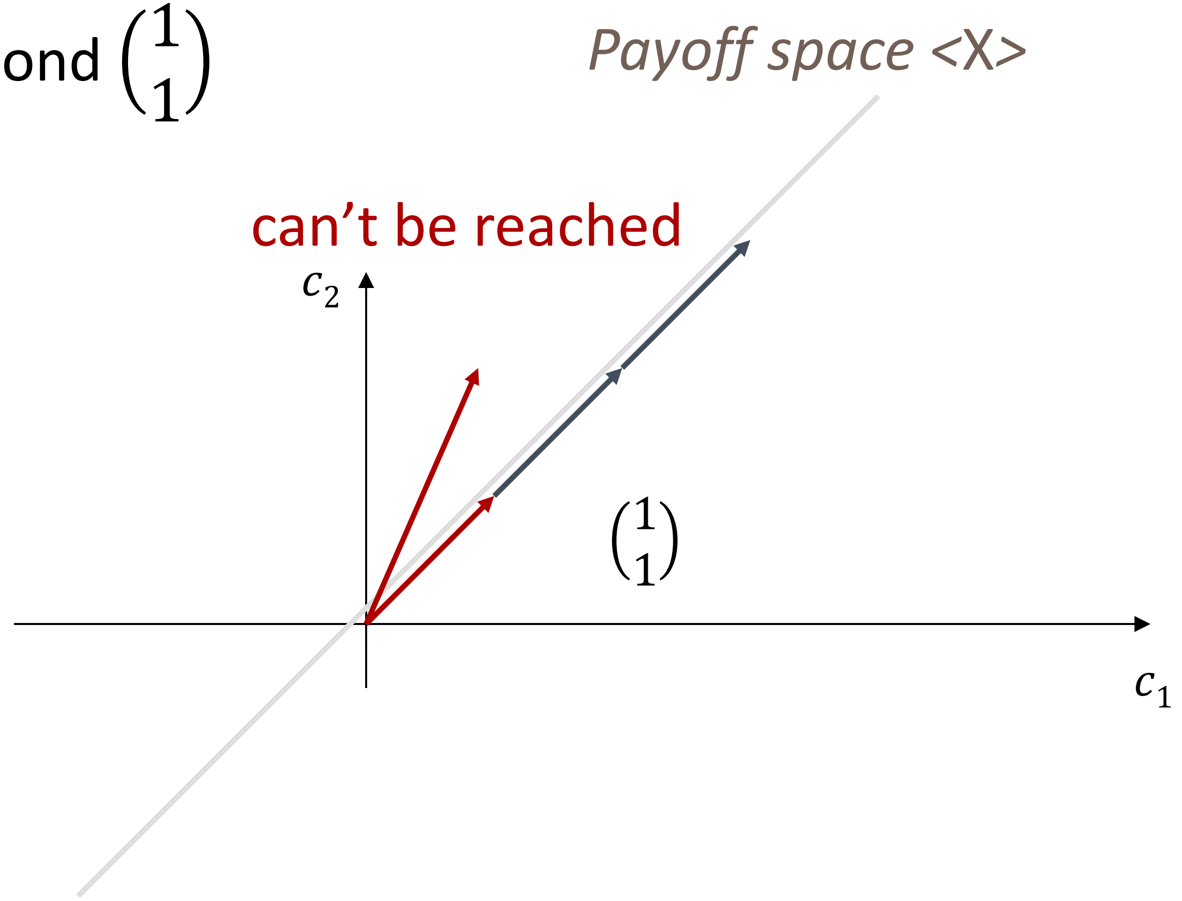
General Security Structure

Only bond $\begin{pmatrix}1\\1\end{pmatrix}$



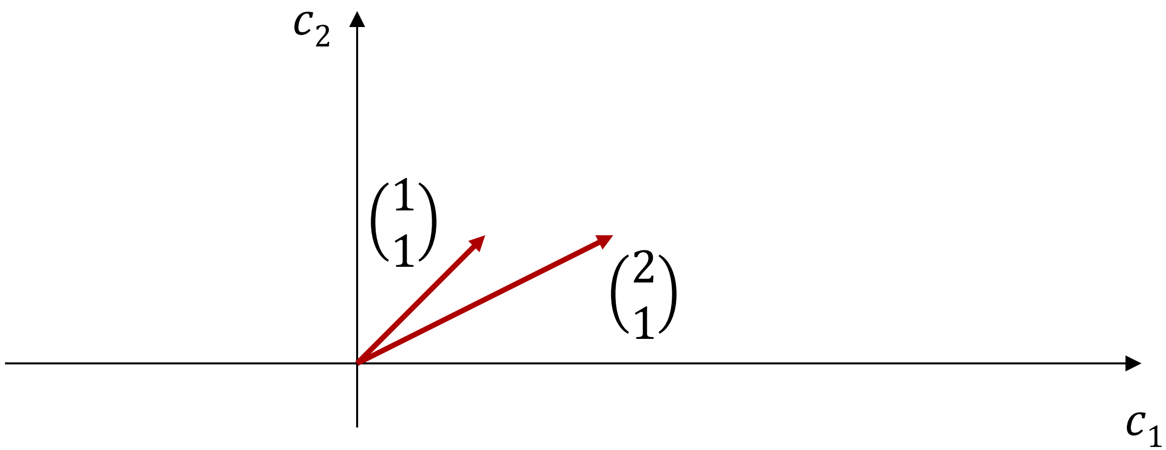
General Security Structure

Only bond $\begin{pmatrix}1\\1\end{pmatrix}$



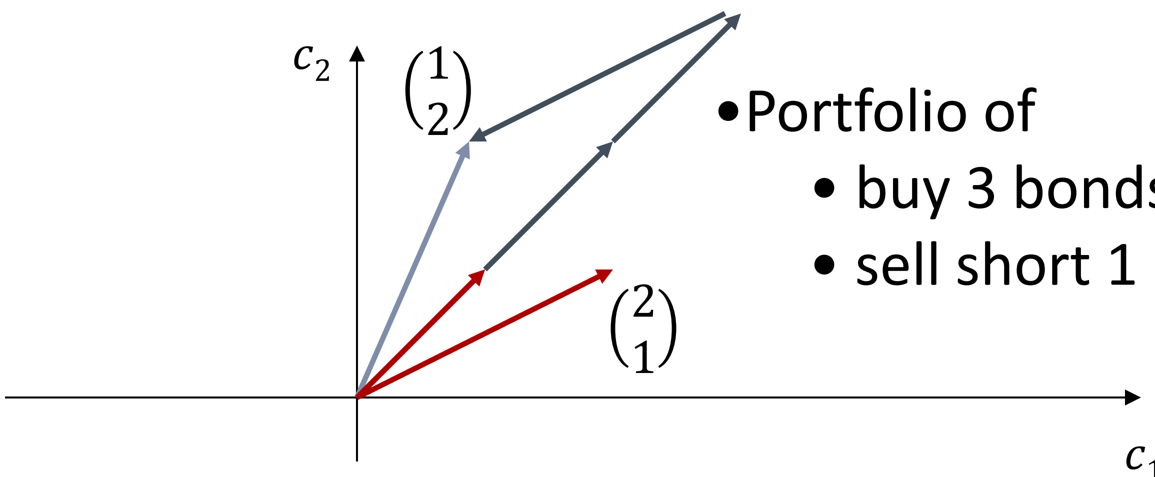
General Security Structure

Add security $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to bond $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$



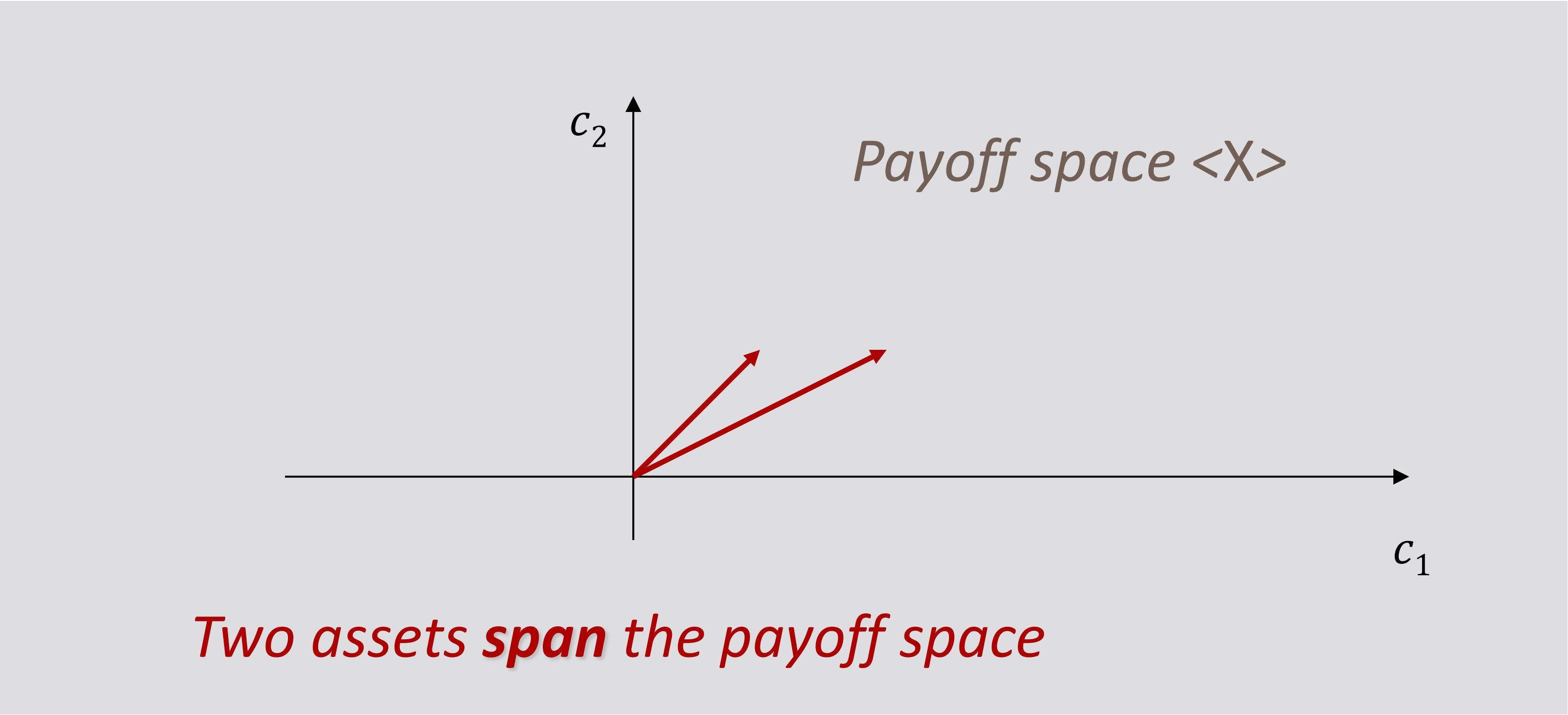
General Security Structure

Add security $\binom{2}{1}$ to bond $\binom{1}{1}$



- Portfolio of
  - buy 3 bonds
  - sell short 1 risky asset

General Security Structure



Market are complete with security structure $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

Payoff space coincides with payoff space of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

General Security Structure

- Portfolio: vector $h \in \mathbb{R}^J$ (quantity for each asset)
- Payoff of Portfolio h is $\sum_j h^j x^j = Xh$
- Asset span
$$\langle X \rangle = \{z \in \mathbb{R}^S : z = Xh \text{ for some } h \in \mathbb{R}^J\}$$
  - $\langle X \rangle$ is a linear subspace of $\mathbb{R}^S$
  - Complete markets $\langle X \rangle = \mathbb{R}^S$
  - Complete markets if and only if $\text{rank}(X) = S$
  - Incomplete markets if $\text{rank}(X) < S$
  - Security j is redundant if $x^j = Xh$ with $h^j = 0$

Introducing derivatives

- Securities: property rights/contracts
- Payoffs of derivatives *derive* from payoff of underlying securities
- Examples: forwards, futures, call/put options

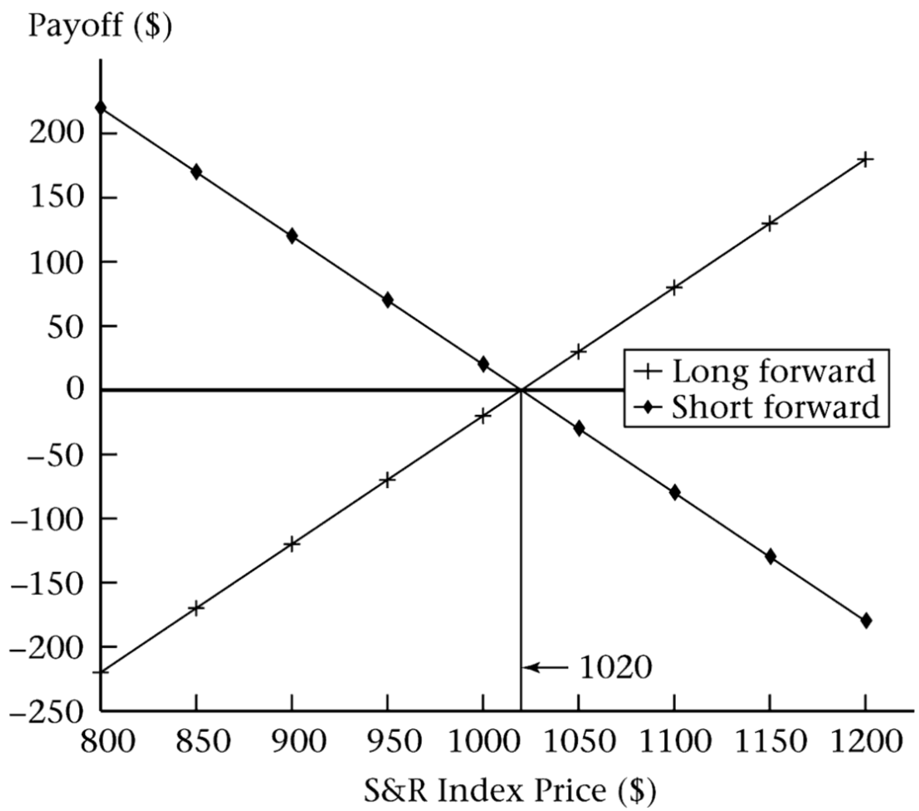
- Question:
  Are derivatives necessarily redundant assets?

Forward contracts

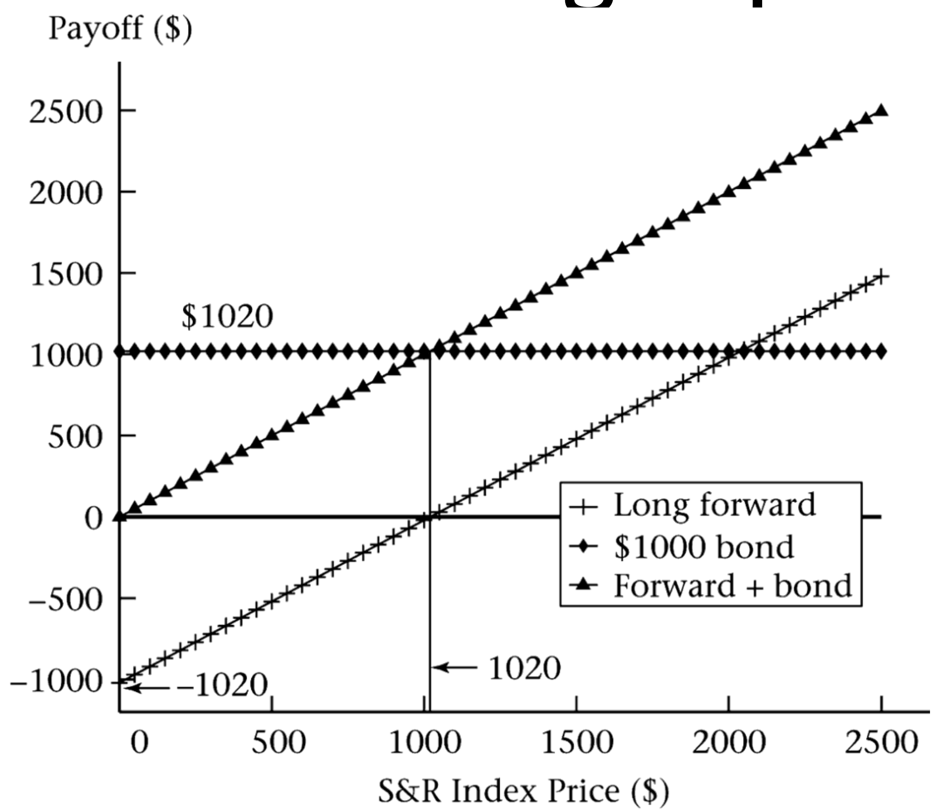
- Definition: A binding agreement (obligation) to buy/sell an underlying asset in the future, at a price set today
- Futures contracts are same as forwards in principle except for some institutional and pricing differences
- A forward contract specifies:
  - The features and quantity of the asset to be delivered
  - The delivery logistics, such as time, date, and place
  - The price the buyer will pay at the time of delivery

Payoff diagram for forwards

- Long and short forward positions on the S&R 500 index:



Forward vs. outright purchase



- Forward + bond = $\overbrace{\text{Spot price at expiration - \$1,020}}^{\text{Forward payoff}}$ + $\overbrace{\text{\$1,020}}^{\text{Bond payoff}}$
  = Spot price at expiration

Additional considerations (ignored)

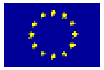
- Type of settlement
  - Cash settlement: less costly and more practical
  - Physical delivery: often avoided due to significant costs
- Credit risk of the counter party
  - Major issue for over-the-counter contracts
    - Credit check, collateral, bank letter of credit
  - Less severe for exchange-traded contracts
    - Exchange guarantees transactions, requires collateral

Call options

- A non-binding agreement (right but not an obligation) to buy an asset in the future, at a price set today
- Preserves the upside potential ( ), while at the same time eliminating the unpleasant ( ) downside (for the buyer)
- The seller of a call option is obligated to deliver if asked

Definition and Terminology

- A call option gives the owner the right but not the obligation to buy the underlying asset at a predetermined price during a predetermined time period
- Strike (or exercise) price: The amount paid by the option buyer for the asset if he/she decides to exercise
- Exercise: The act of paying the strike price to buy the asset
- Expiration: The date by which the option must be exercised or become worthless
- Exercise style: Specifies when the option can be exercised


European-style: can be exercised only at expiration date


American-style: can be exercised at any time before expiration

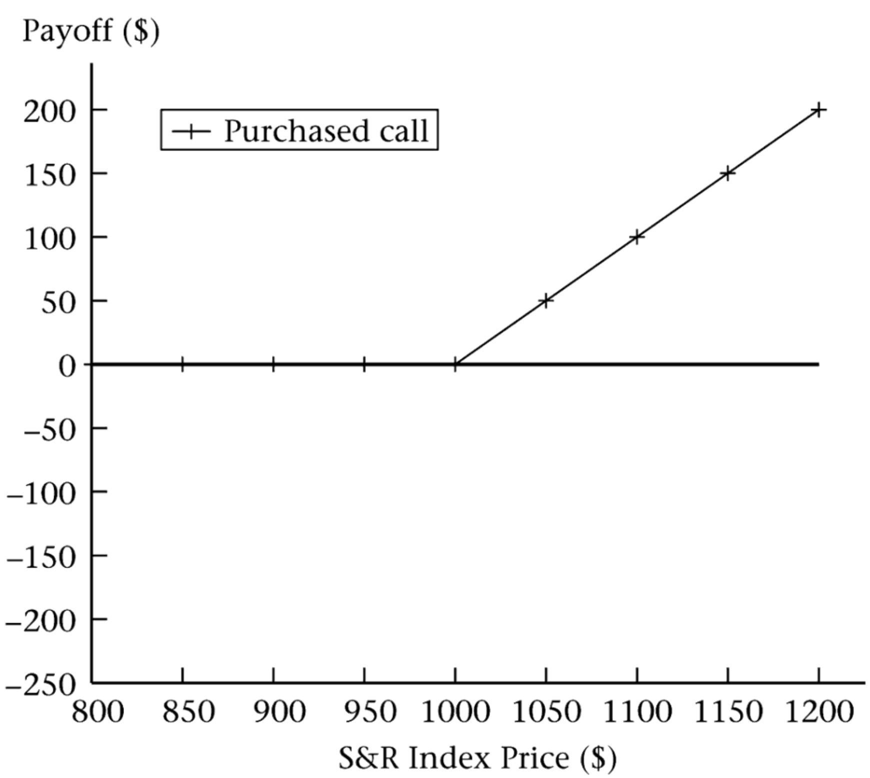

Bermudan-style: can be exercised during specified periods

Payoff/profit of a purchased call

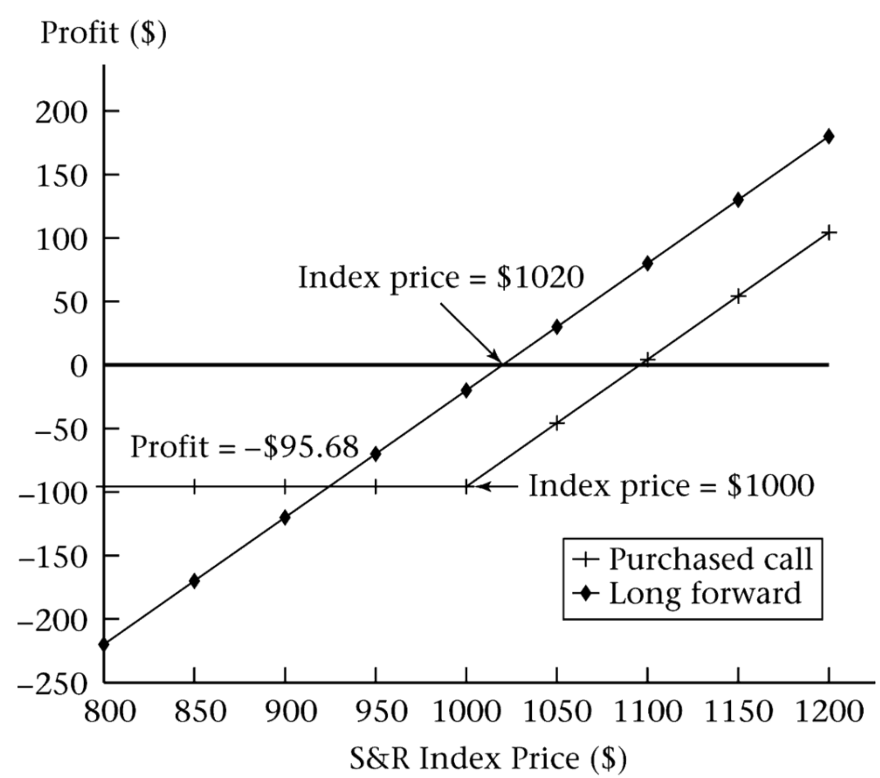
- Payoff = *max* [0, spot price at expiration – strike price]
- Profit = Payoff – future value of option premium
- Examples 2.5 & 2.6:
  - S&R Index 6-month Call Option
    - Strike price = $1,000, Premium = $93.81, 6-month risk-free rate = 2%
  - If index value in six months = $1100
    - Payoff = *max* [0, $1,100 - $1,000] = $100
    - Profit = $100 – ($93.81 x 1.02) = $4.32
  - If index value in six months = $900
    - Payoff = *max* [0, $900 - $1,000] = $0
    - Profit = $0 – ($93.81 x 1.02) = - $95.68

Diagrams for purchased call

- Payoff at expiration



- Profit at expiration



Put options

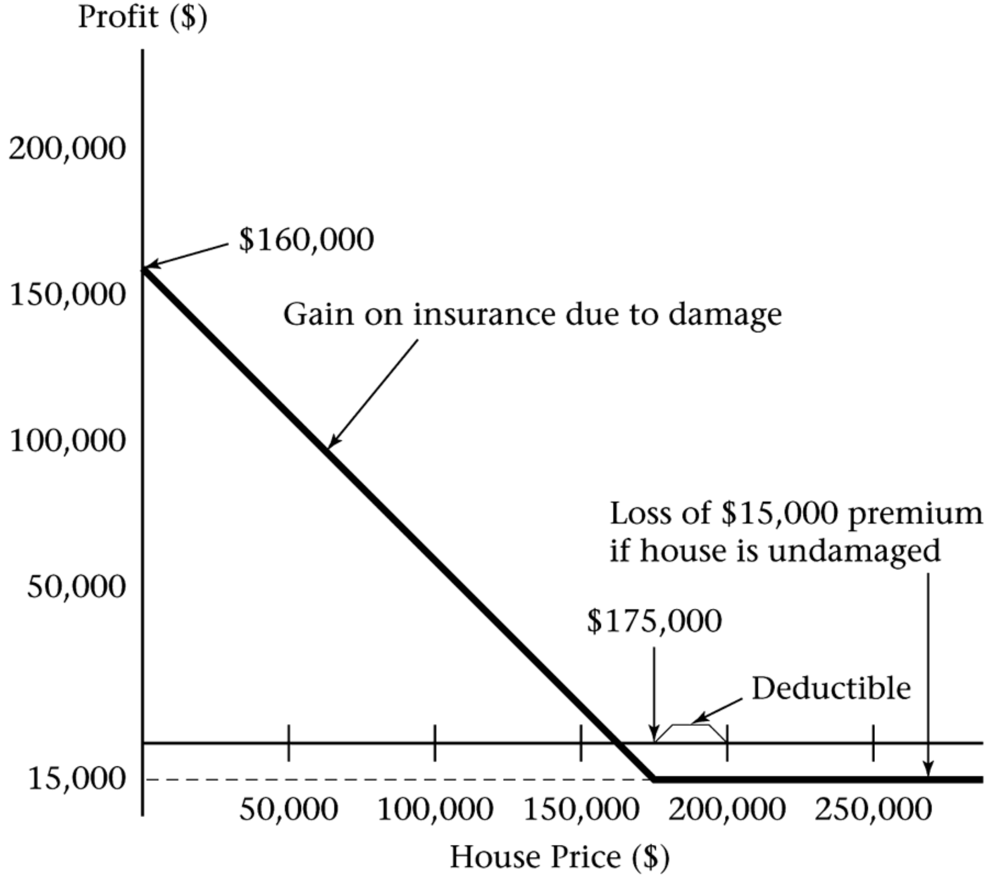
- A put option gives the owner the right but not the obligation to sell the underlying asset at a predetermined price during a predetermined time period
- The seller of a put option is obligated to buy if asked
- Payoff/profit of a purchased (i.e., long) put:
  - Payoff = *max* [0, strike price – spot price at expiration]
  - Profit = Payoff – future value of option premium
- Payoff/profit of a written (i.e., short) put:
  - Payoff = - *max* [0, strike price – spot price at expiration]
  - Profit = Payoff + future value of option premium

A few items to note

- A call option becomes more profitable when the underlying asset appreciates in value
- A put option becomes more profitable when the underlying asset depreciates in value
- Moneyness:
  - In-the-money option: positive payoff if exercised immediately
  - At-the-money option: zero payoff if exercised immediately
  - Out-of-the money option: negative payoff if exercised immediately

Options and insurance

- Homeowner's insurance as a put option:



Equity linked CDs

- The 5.5-year CD promises to repay initial invested amount and 70% of the gain in S&P 500 index:
  - Assume $10,000 invested when S&P 500 = 1300
  - Final payoff =

$$\$10000 \times \left(1 + 0.7 \times \max\left\{0, \frac{S_{final}}{1300} - 1\right\}\right)$$

  - where S_{final}= value of the S&P 500 after 5.5 years

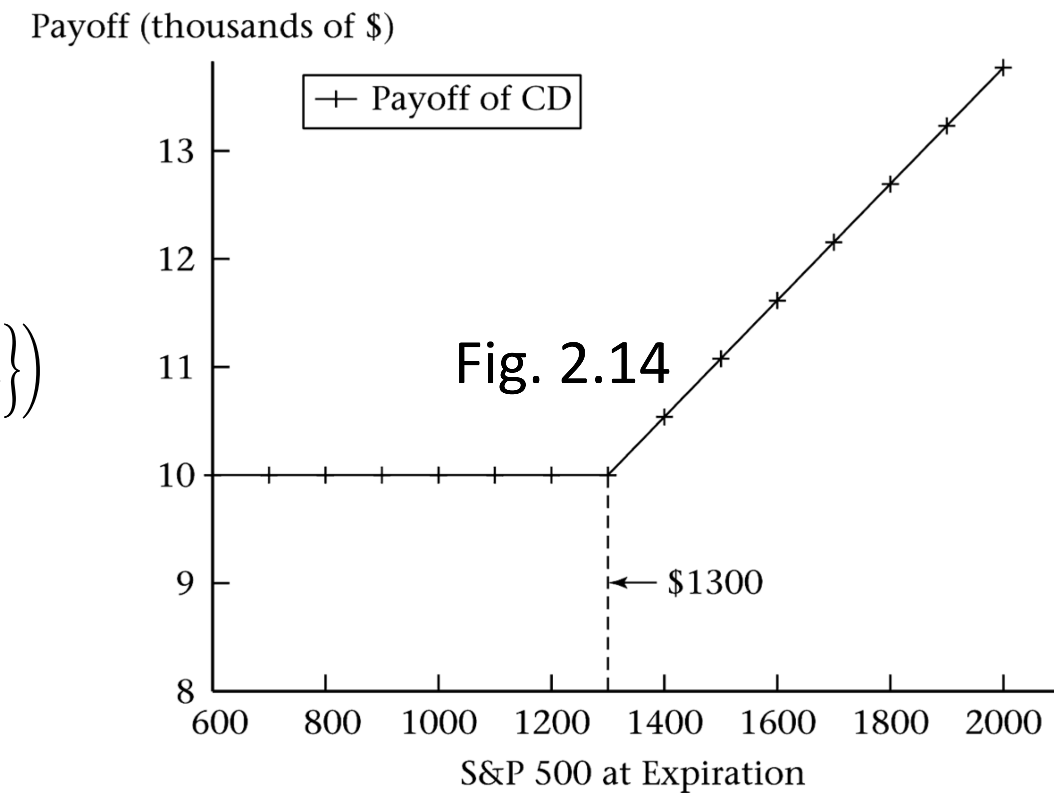
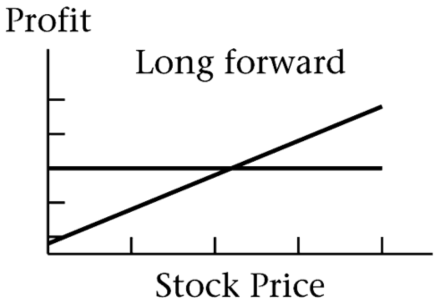
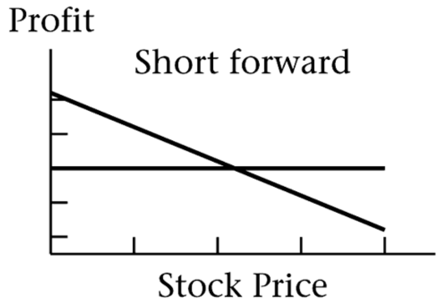


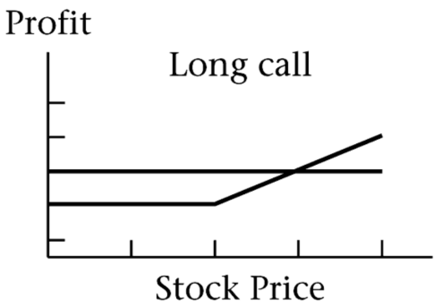
Fig. 2.14

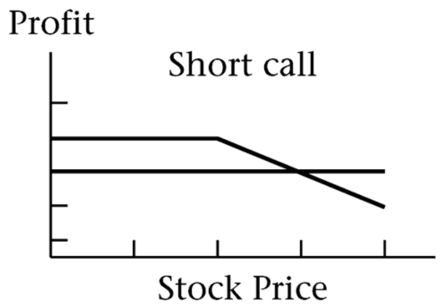
Option and forward positions

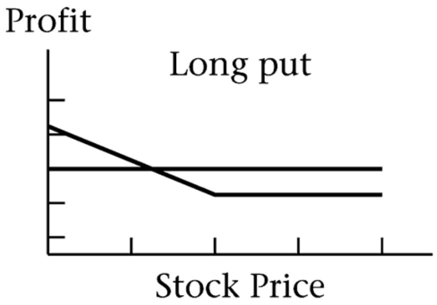
A summary

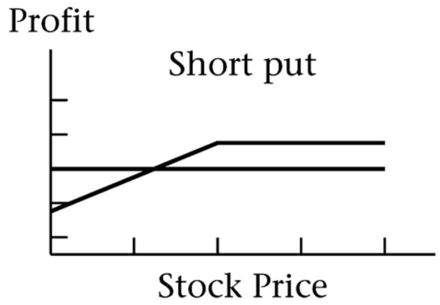












Options to Complete the Market

- Stock's payoff: $x^j = (1,2,\ldots S)'$ (= state space)
- Introduce call options with final payoff at T:

$$C_T = \max\{S_T - K, 0\} = [S_T - K]^+$$

- Thus

$$C_{K=1} = (0,1,2,\ldots,S-2,S-1)'$$
$$C_{K=2} = (0,0,1,\ldots,S-3,S-2)'$$
$$\vdots$$
$$C_{K=S-1} = (0,0,0,\ldots,0,1)'$$

Options to Complete the Market

- Together with the primitive asset:

$$X = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 3 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ S-2 & S-3 & S-4 & \cdots & 1 & 0 & 0 \\ S-1 & S-2 & S-3 & \cdots & 2 & 1 & 0 \\ S & S-1 & S-2 & \cdots & 3 & 2 & 1 \end{pmatrix}$$

Homework: check whether markets are complete

General Security Structure

- Price vector $p \in \mathbb{R}^J$ of asset prices
- Cost of portfolio h,

$$p \cdot h := \sum_j p^j h^j$$

- If $p^j \neq 0$ the (gross) return vector of asset *j* is the vector

$$R^j = \frac{x^j}{p^j}$$