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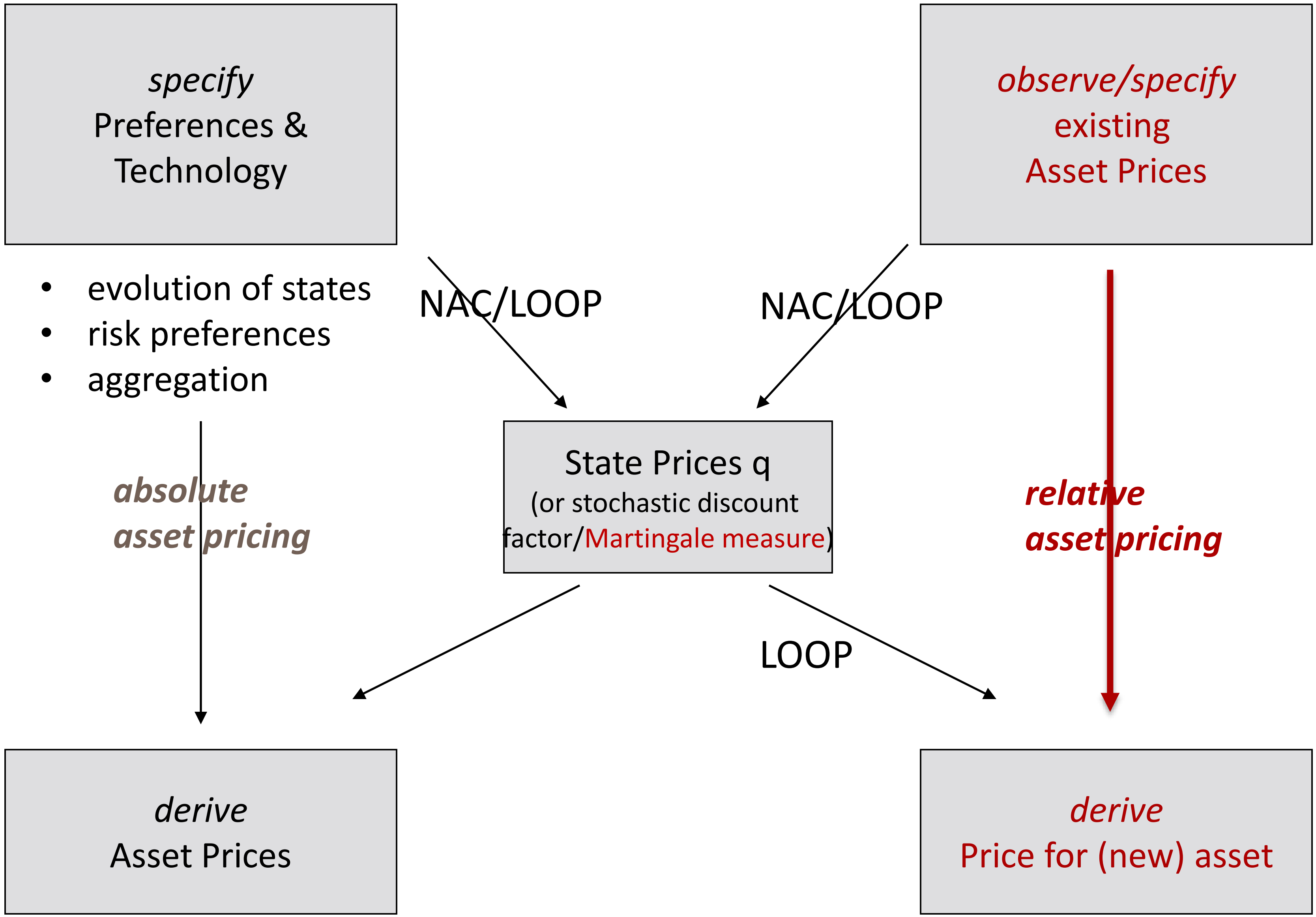
LECTURE 08: MULTI-PERIOD MODEL OPTIONS: BLACK-SCHOLES-MERTON



Only works as long as market completeness doesn't change

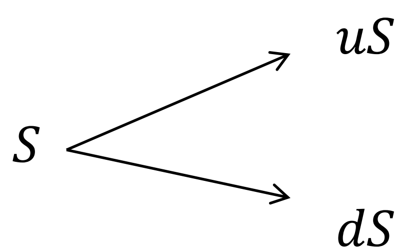
- Price claim at $t = 0$ with many final payoffs at T
  - S states/histories
- Dynamic replication with only 2 assets
  - Dynamic trading strategy

Option Pricing

- European call option maturing at time T with strike $K \Rightarrow C_T = (S_T - K)^+$, no cash flows in between
- Why multi-period problem
  - Not able to statically replicate this payoff using just the stock and risk-free bond
  - Need to dynamically hedge – required stock position changes for each period until maturity
    - Recall static hedge for forward, using put-call parity
- Replication strategy depends on specified random process need to know how stock price evolves over time.
  - Binomial (Cox-Ross-Rubinstein) model is canonical

Binominal Option Pricing

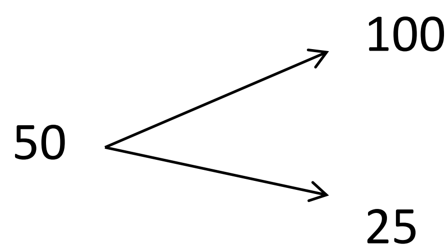
- Assumptions:
  - Bond:
    - Constant risk-free rate $R^f = R_t^f \ \forall t$
      $R^f = e^{rT/n}$ for period with length $^T/_n$
  - Stock:
    - which pays no dividend
    - stock price moves from S to either uS or dS,



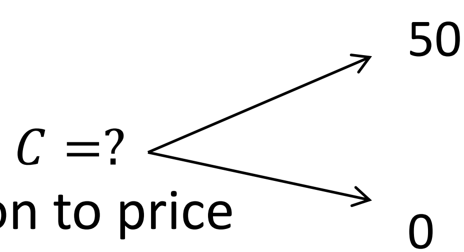
    - i.i.d. over time ⇒ final distribution of S_T is binomial
    - No arbitrage: $u > R^f > d$

A One-Period Binomial Tree

- Example of a single-period model
  - $S = 50, u = 2, d = 0.5, R^f = 1.25$



  - What is value of a European call option with $K = 50$?
  - Option payoff: $(S_T - K)^+$



  - Use replication to price

Single-period Replication

- Long Δ stocks and B dollars in bond
- Payoff from portfolio:

$$\Delta S + B = 50\Delta + B \quad \begin{cases} \nearrow \; \Delta u S + R^f B = 100\Delta + 1.25B \\ \searrow \; \Delta d S + R^f B = 25\Delta + 1.25B \end{cases}$$

- C_u option payoff in up state and C_d as option payoff in down state

$$C_u = 50, C_d = 0$$

- Replicating strategy must match payoffs:

$$C_u = \Delta u S + R^f B$$
$$C_d = \Delta d S + R^f B$$

Single-period Replication

- Solving these equations yields:

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

$$B = \frac{(uC_d - dC_u)}{R^F(u-d)}$$

- In previous example, $\Delta = \frac{2}{3}$ and $B = -13.33$, so the option value is

$$C = \Delta S + B = 20$$

- Interpretation of Δ: sensitivity of call price to a change in the stock price. Equivalently, tells you how to hedge risk of option
  - To hedge a long position in call, short Δ shares of stock

Risk-neutral Probabilities

- Substituting Δ and B from into formula

$$C = \frac{C_u - C_d}{S(u-d)} S + \frac{uC_d - dC_u}{R^f(u-d)} = \frac{1}{R^f}\left[\underbrace{\frac{R^f - d}{u-d}}_{\pi_u^Q} C_u + \underbrace{\frac{u - R^f}{u-d}}_{1-\pi_u^Q} C_d\right]$$

where π^Q is risk neutral prob. (EMM)

- Hence, like any asset $p^j = \frac{1}{R^f} E^Q[x^j]$

$$C = \frac{1}{R^f}\left[\pi_u^Q C_u + \left(1 - \pi_u^Q\right) C_d\right] = \frac{1}{R^f} E^Q[C_{t+1}]$$

Risk-neutral Probabilities

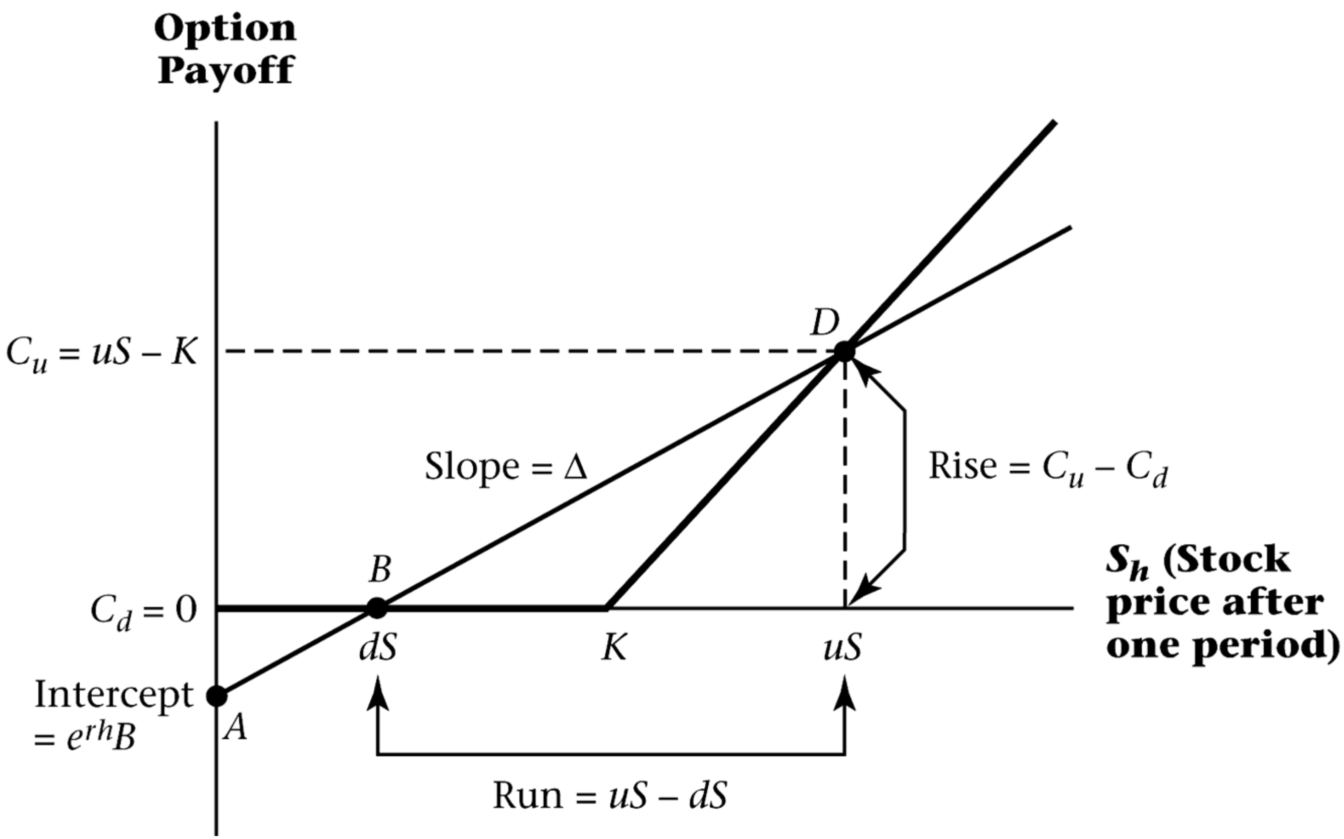
- Note that π^Q is the probability that would justify the current stock price S in a risk-neutral world:

$$S = \frac{1}{R^f}\left[\pi_u^Q uS + \left(1 - \pi_u^Q\right) dS\right]$$

$$\pi_u^Q = \frac{R^f - d}{u - d}$$

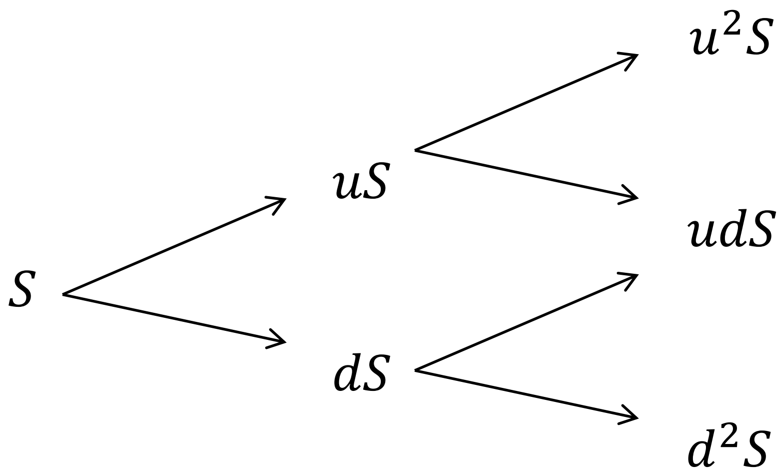
- No arbitrage requires $u > R^f > d$
- Note: relative asset pricing
  - we don't need to know objective probability (P-measure).
  - Q-measure is sufficient

The Binomial Formula in a Graph



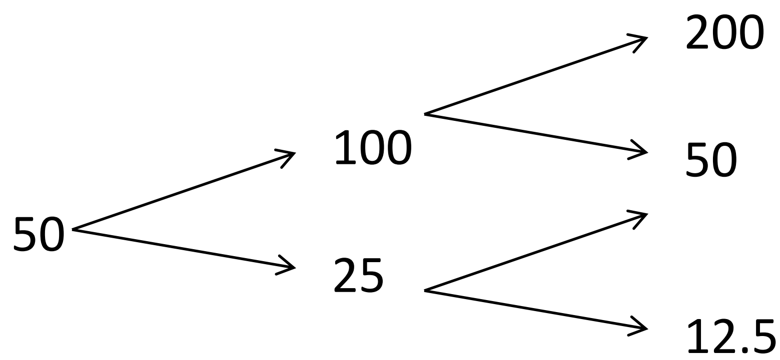
Two-period Binomial Tree

- Concatenation of single-period trees:

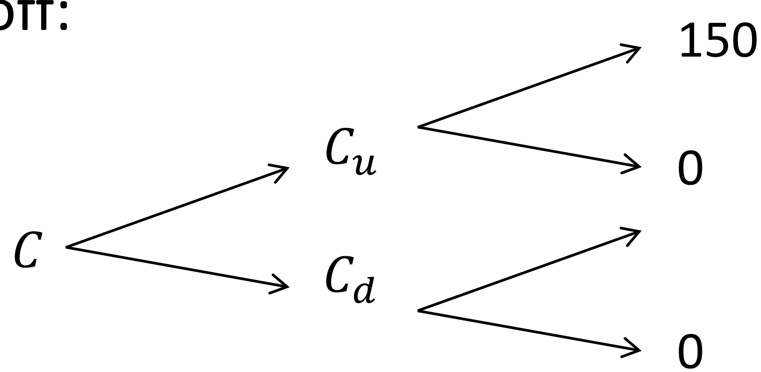


Two-period Binomial Tree

- Example: $S = 50, u = 2, d = 0.5, R^f = 1.25$

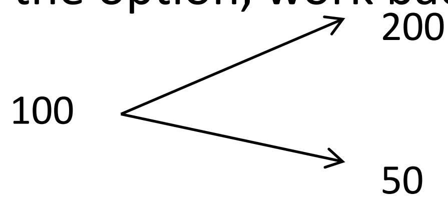


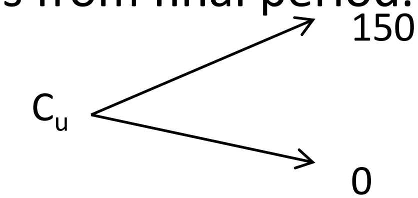
- Option payoff:



Two-period Binomial Tree

- To price the option, work backwards from final period.





- We know how to price this from before:

$$\pi_u^Q = \frac{R^f - d}{u - d} = \frac{1.25 - 0.5}{2 - 0.5} = 0.5$$

$$C_u = \frac{1}{R^f}\left[\pi_u^Q C_{uu} + \left(1 - \pi_u^Q\right) C_{ud}\right] = 60$$

- **Three-step procedure:**
  1. Compute risk-neutral probability, π_u^Q
  2. Plug into formula for C at each node for prices, going backwards from final node.
  3. Plug into formula for Δ and B at each node for replicating strategy, going backwards from the final node..

Two-period Binomial Tree

- General formulas for two-period tree:
- $\pi_u^Q = (R-d)/(u-d)$

$$C = \frac{\pi_u^Q C_u + (1-\pi_u^Q) C_d}{R^f}$$

$$= \frac{\pi_u^{Q\,2} C_{uu} + 2\pi_u^Q \left(1-\pi_u^Q\right) C_{ud} + \left(1-\pi_u^Q\right)^2 C_{ud}}{(R^f)^2}$$

$$\Delta = \frac{C_u - C_d}{uS - dS}, B = C - \Delta S$$

$$C_u = \frac{\pi_u^Q C_{uu} + (1-\pi_u^Q) C_{ud}}{R^f}$$

$$\Delta_u = \frac{C_{uu} - Cud}{u^2 S - udS}$$

$$B_u = C_u - \Delta_u S$$

$$C_d = \frac{\pi_u^Q C_{ud} + (1-\pi_u^Q) C_{dd}}{R^f}$$

$$\Delta_d = \frac{C_{ud} - C_{dd}}{udS - d^2 S}$$

$$B_d = C_d - \Delta_d S$$

C_{uu}

C_{ud}

C_{dd}

- Synthetic option requires ***dynamic hedging***
  - Must change the portfolio as stock price moves

Arbitraging a Mispriced Option

- 3-period tree:
  - $S = 80, K + 80, u = 1.5, d = 0.5, R = 1.1$
- Implies $\pi_u^Q = \frac{R^f - d}{u - d} = 0.6$
- Cost of dynamic replication strategy $34.08
- If the call is selling for $36, how to arbitrage?
  - Sell the real call
  - Buy the synthetic call
- Up-front profit:
  - $C - \Delta S + B = 36 – 34.08 = 1.92$

Towards Black-Scholes

- General binomial formula for a European call on non-dividend paying stock n periods from expiration:

$$C = \frac{1}{(R^f)^n}\left[\sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!} \left(\pi_u^Q\right)^j \left(1-\pi_u^Q\right)^{n-j} \left(u^j d^{n-j} S - K\right)^+\right]$$

- Take parameters:

$$u = e^{\sigma\sqrt{\frac{T}{n}}}, d = \frac{1}{u} = e^{-\sigma\sqrt{\frac{T}{n}}}$$

- Where:
  - n = number of periods in tree
  - T = time to expiration (e.g., measured in years)
  - σ = standard deviation of continuously compounded return
  - Also take $R^f = e^{\frac{r^f T}{n}}$
- As $n \to \infty$
  - Binominal tree → geometric Brownian motion
  - Binominal formula → Black Scholes Merton

Black-Scholes

$$C = S\mathcal{N}(d_1) - Ke^{-r^f T}\mathcal{N}(d_2)$$
$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left[\frac{S}{K}\right] + \left(r^f + \frac{\sigma^2}{2}\right)T\right]$$
$$d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left[\frac{S}{K}\right] + \left(r^f - \frac{\sigma^2}{2}\right)T\right] = d_1 - \sigma\sqrt{T}$$

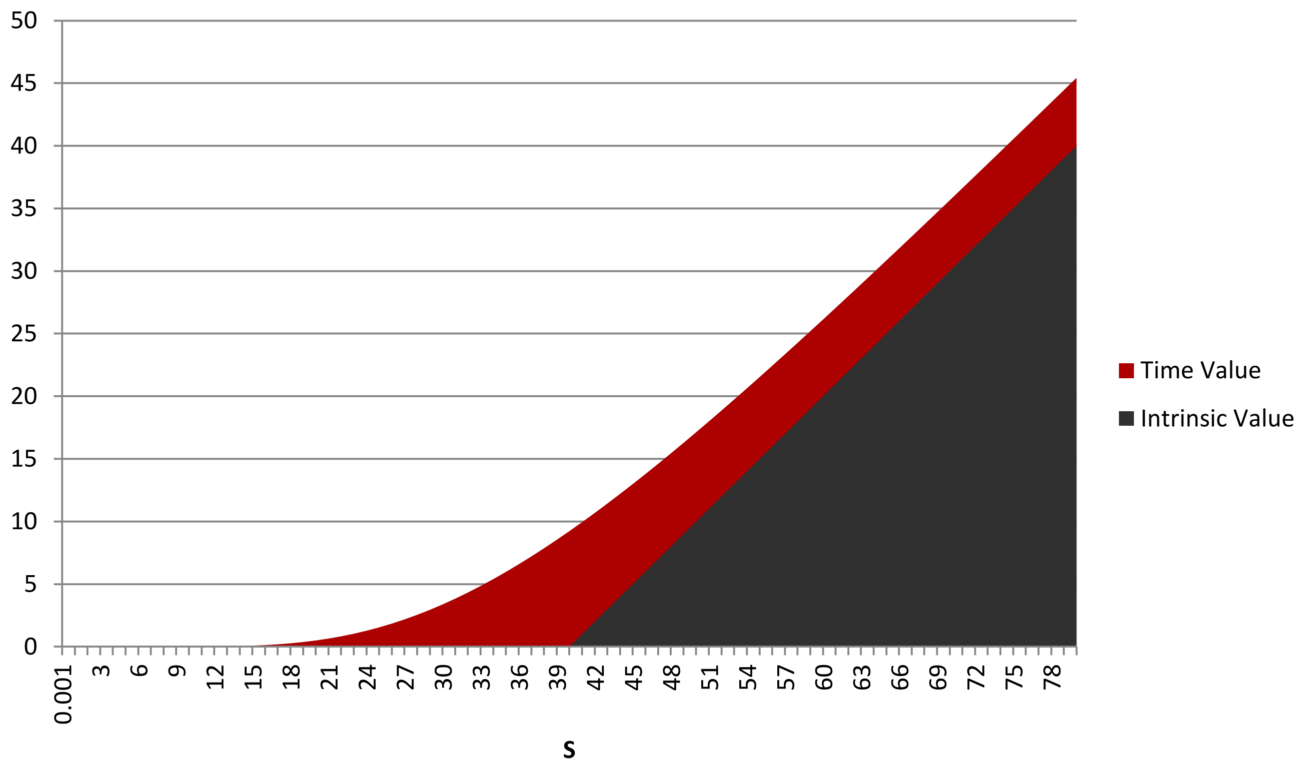
- $S =$ current stock price
- $K =$ strike
- $T =$ time to maturity
- $r =$ annualized continuously compounded risk-free rate,
- $\sigma =$ annualized standard dev. of cont. comp. rate of return on underlying

- Price of a put-option by put-call parity

$$\begin{aligned} P &= C - S + Ke^{-rT} \\ &= Ke^{-rT}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1) \end{aligned}$$

Interpreting Black-Scholes

- Option has ***intrinsic value*** $(S-K)^{+}$ and ***time-value*** $C-(S-K)^{+}$



Delta

- Δ is the sensitivity of option price to a small change in the stock price
  - Number of shares needed to make a synthetic call
  - Also measures riskiness of an option position

$$\Delta_c = \mathcal{N}(d_1)$$
$$B_c = -Ke^{rT}\mathcal{N}(d_2)$$

NB: For $S = K$ and $T \to 0$, $\Delta = \mathcal{N}(d_1) = \frac{1}{2}$

- Delta of
  - Call: $\Delta_c \in [0,1]$
  - Put: $\Delta_c \in [-1,0]$
  - Stock: 1
  - Bond: 0
  - portfolio: $\sum_j h^j \Delta_i$

Option Greeks

- What happens to option price when *one* input changes?
  - Delta (Δ): change in option price when stock increases by $1
  - Gamma (Γ): change in delta when option price increases by $1
  - Vega: change in option price when volatility increases by 1%
  - Theta (θ): change in option price when time to maturity decreases by 1 day
  - Rho (ρ): change in option price when interest rate rises by 1%
- Greek measures for portfolios
  - The Greek measure of a portfolio is weighted average of Greeks of individual portfolio components:

Delta-Hedging

- Portfolio h is delta-neutral if $\sum_j h_j \Delta_j = 0$
- Delta-neutral portfolios are a way to hedge out the risk of an option (or portfolio of options)
  - Example: suppose you write 1 European call whose delta is 0.61. How can you trade to be delta-neutral?

$$h_c \Delta_c + h_s \Delta_s = -1(0.61) + h_s(1) = 0$$

  - So we need to hold 0.61 shares of the stock.
- Delta hedging makes you directionally neutral on the position.
  - But only linearly! - Γ

Portfolio Insurance & 1987 Crash

- Δ-replication strategy leads to inverse demand
  - Sell when price goes down
  - Buy when price goes up
- $\Rightarrow$ Destabilizes market
- 1987 – (uninformed) portfolio insurance trading was interpreted as “informed” sellers.

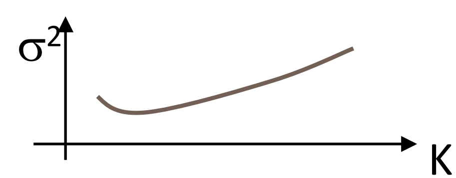
Notes on Black-Scholes

- Delta-hedging is not a perfect hedge if you do not trade continuously
  - Delta-hedging is a linear approximation to the option value
  - But convexity implies second-order derivatives matter
  - Hedge is more effective for smaller price changes
- Delta-Gamma hedging reduces the basis risk of the hedge.
- B-S model assumes that volatility is constant over time and returns are normal – no fat tails.
  - Volatility “smile”
  - BS underprices out-of-the-money puts (and thus in-the-money calls)
  - BS overprices out-of-the-money calls (and thus in-the-money puts)
  - Ways forward: stochastic volatility
- Other issues: stochastic interest rates, bid-ask transaction costs, etc.

Implied Vol., Smiles and Smirks

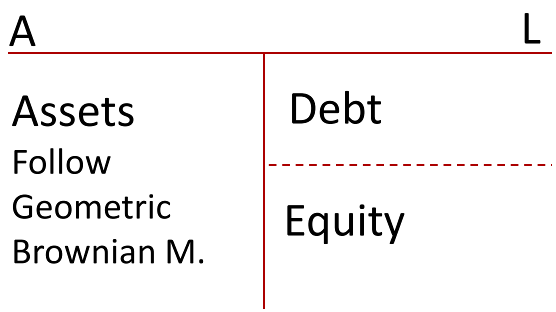
- Implied volatility
  - Use current option price and assume B-S model holds
  - Back out volatility
  - VIX versus implied volatility of 500 stocks
- Smile/Smirk
  - Implied volatility across various strike prices
    - BS implies horizontal line
  - Smile/Smirk after 1987

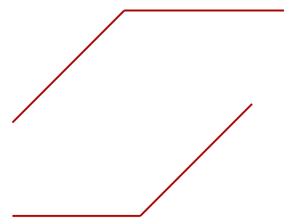
Smile/smirk



Merton Model

- Firm's balance sheet





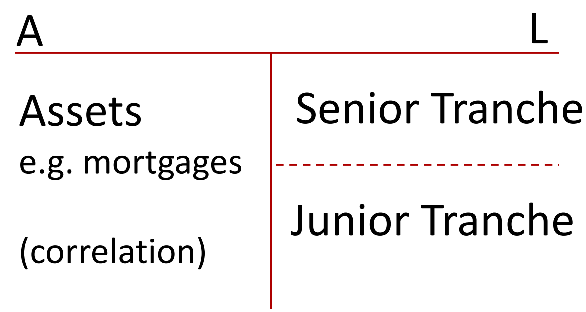
Put option payoff (short)

Call option payoff (long)

  - Call on equity is essential a "call on a call"
- Merton: Observe equity prices (including vol) and back out
  - firm value
  - debt value

Collateral Debt Obligations (CDO)

- Collateralized Debt Obligation- repackage cash flows from a set of assets



- Tranches:
  - Senior tranche paid out first,
  - Mezzanine second,
  - junior tranche is paid out last
- Can adapt option pricing theory, useful in pricing CDOs:
  - Tranches can be priced using analogues from option pricing formulas
  - Estimate “implied default correlations” that price the tranches correctly

Pricing of Any Non-linear Payoff

- Method can be used to price any non-linear payoff