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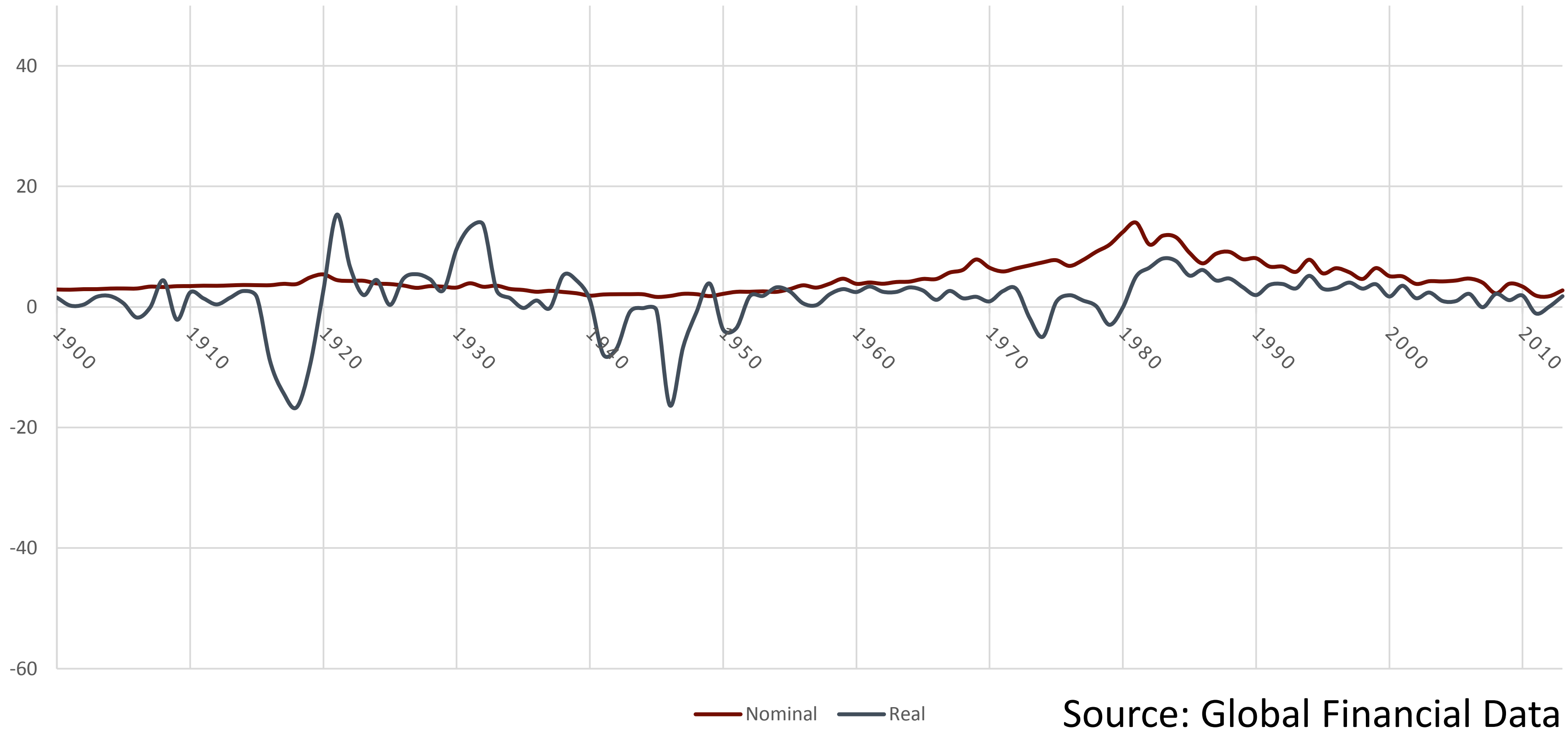
LECTURE 09: MULTI-PERIOD MODEL BONDS

Overview

1. Bond Basics
2. Term Structure
   - Expectations Hypothesis
   - Canonical Term Structure Models
3. Duration
4. Repos

U.S. Treasury

– Bills (< 1 year), no coupons, sell at discount
– Notes (1-10 years), Bonds (10-30 years), coupons, sell at par (10 year)

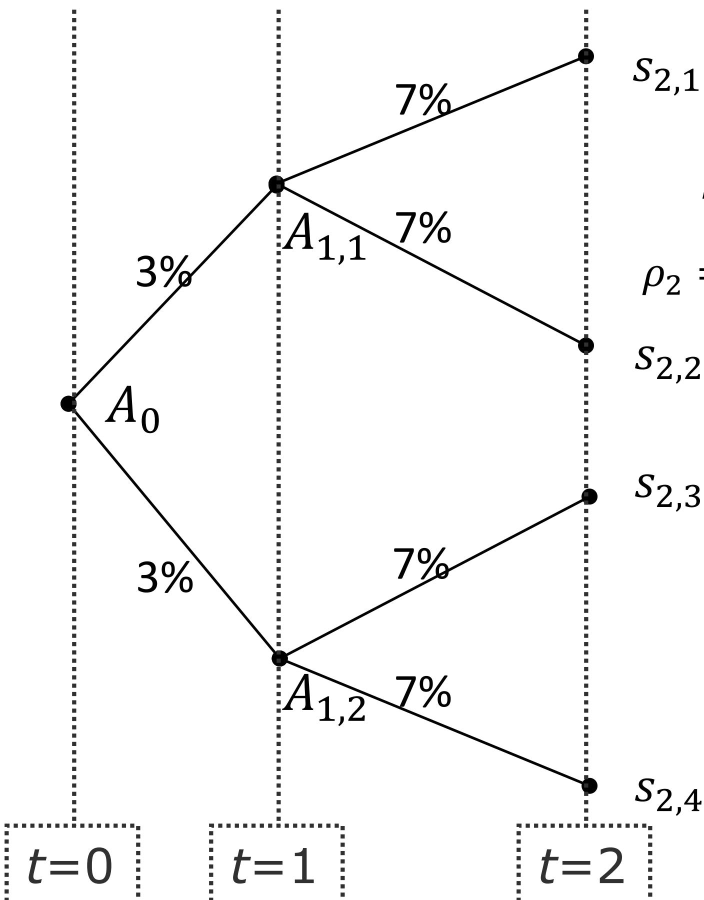


Source: Global Financial Data

Bond Basics

- Notation:
  - $r_t(t_1, t_2)$: Interest rate from time t_1 to t_2 prevailing at time t.
    - Spot (short) rate: $t_1 = t$ and $t_2 = t + 1$
    - Forward rate
  - $B_t(t_1, t_2, c_\tau)$: Bond price quoted at t to be purchased at t_1 maturing at t_2 with coupon payments c_τ at various τ
  - $Z_t(t_1, t_2)$: Price of a zero coupon bond, only pays at time t_2
  - $y_t^{(N)}$: Yield at time t for a bond maturing in $t_2 = t + N$
    - Just a different way to quote bond price
    - Yield to maturity: Constant discount rate at which the sum of the discounted future cash flows (coupons and principal) is equal to the price of the bond

Deterministic vs. Stochastic Rate

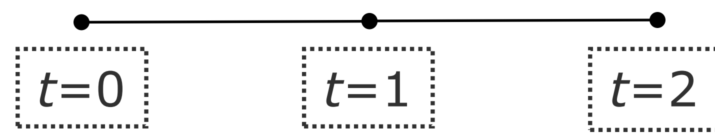


$$\rho_1 = \frac{1}{1.03}$$

$$\rho_2 = \frac{1}{1.03 * 1.07}$$

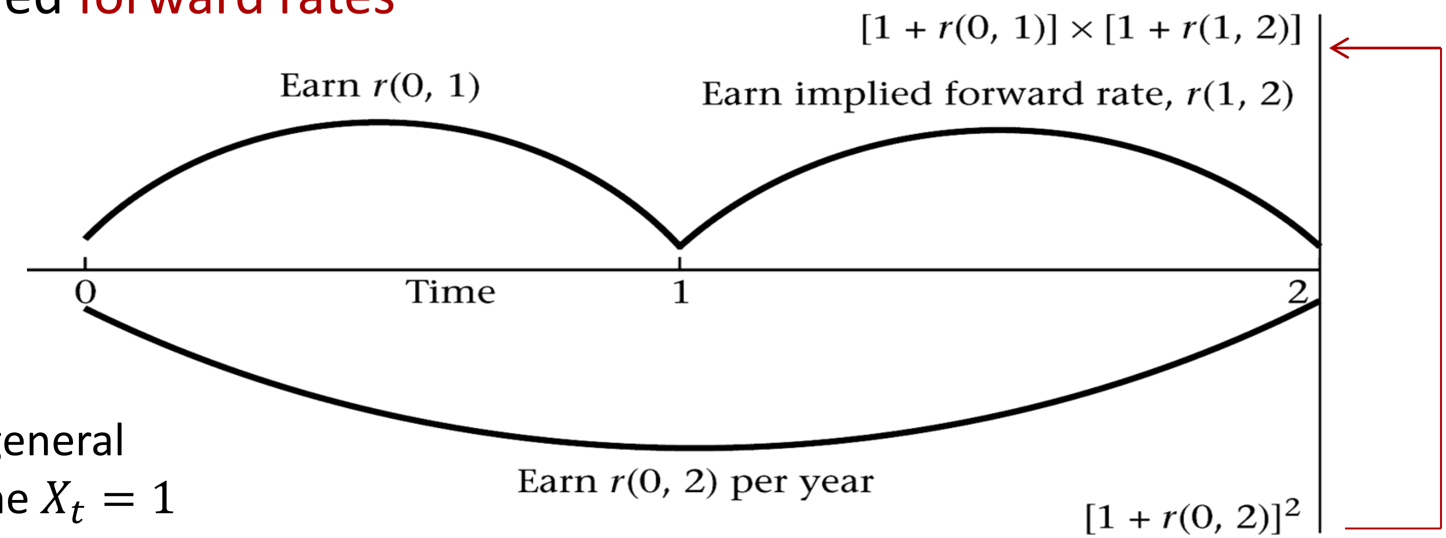
If only bond prices matter
(and other assets can be ignored)

Can simplify event tree to



Bond Basics under Certainty

- Price of a Zero-coupon bond that pays X_t: $Z_0(0,t) = \frac{X_t}{\left(1+y_0^{(t)}\right)^t}$
- Yield curve: annualized bond yields $r_0(0,t) = y_0^{(t)}$
- Implied forward rates



Note: in general we assume $X_t = 1$

- $[1 + r_0(t_1, t_2)]^{t_2 - t_1} = \frac{(1+r_0(0,t_2))^{t_2}}{(1+r_0(0,t_1))^{t_1}} = \frac{Z_0(0,t_1)}{Z_0(0,t_2)}$

Bond Basics (cont.)

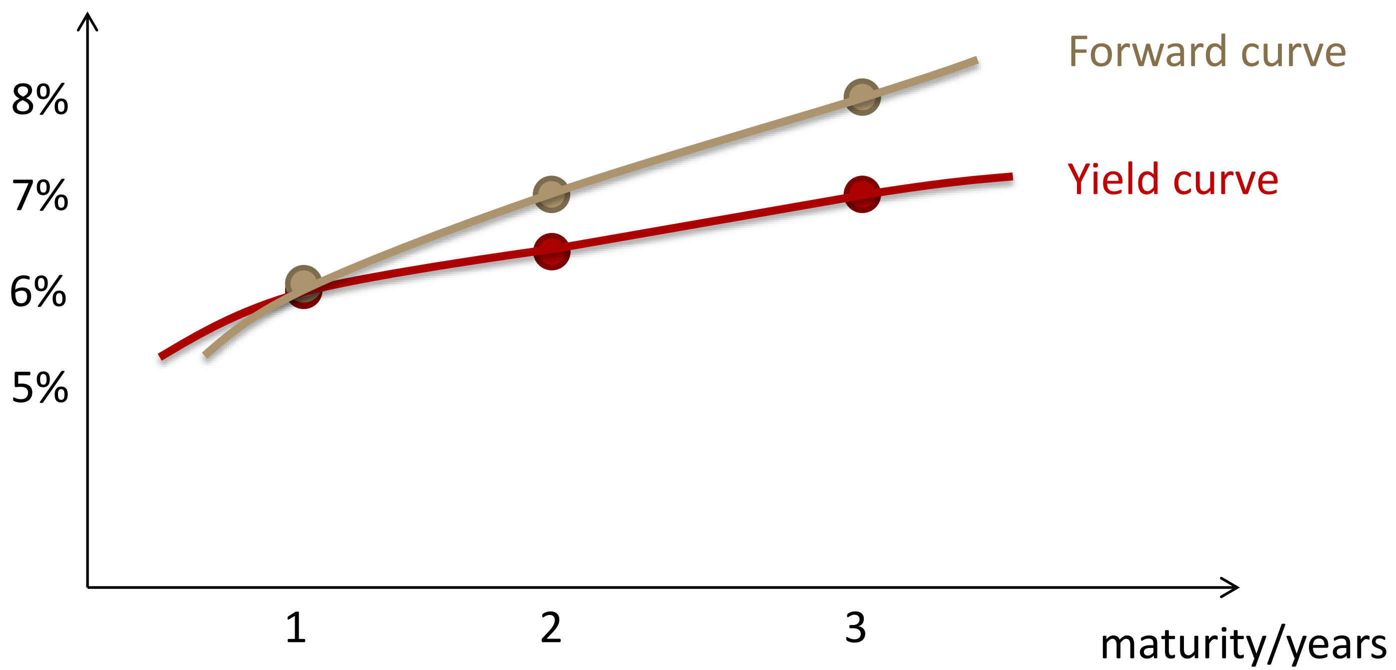
- Zero-coupon bonds make a single payment at maturity

TABLE 7.1 Five ways to present equivalent information about default-free interest rates. All rates but those in the the last column are effective annual rates.

Maturity	(1) Zero-Coupon Bond Yield	(2) Zero-Coupon Bond Price	(3) One-Year Implied Forward Rate	(4) Par Coupon	(5) Continuously Compounded Zero Yield
1	6.00%	0.943396	6.00000%	6.00000%	5.82689%
2	6.50	0.881659	7.00236	6.48423	6.29748
3	7.00	0.816298	8.00705	6.95485	6.76586

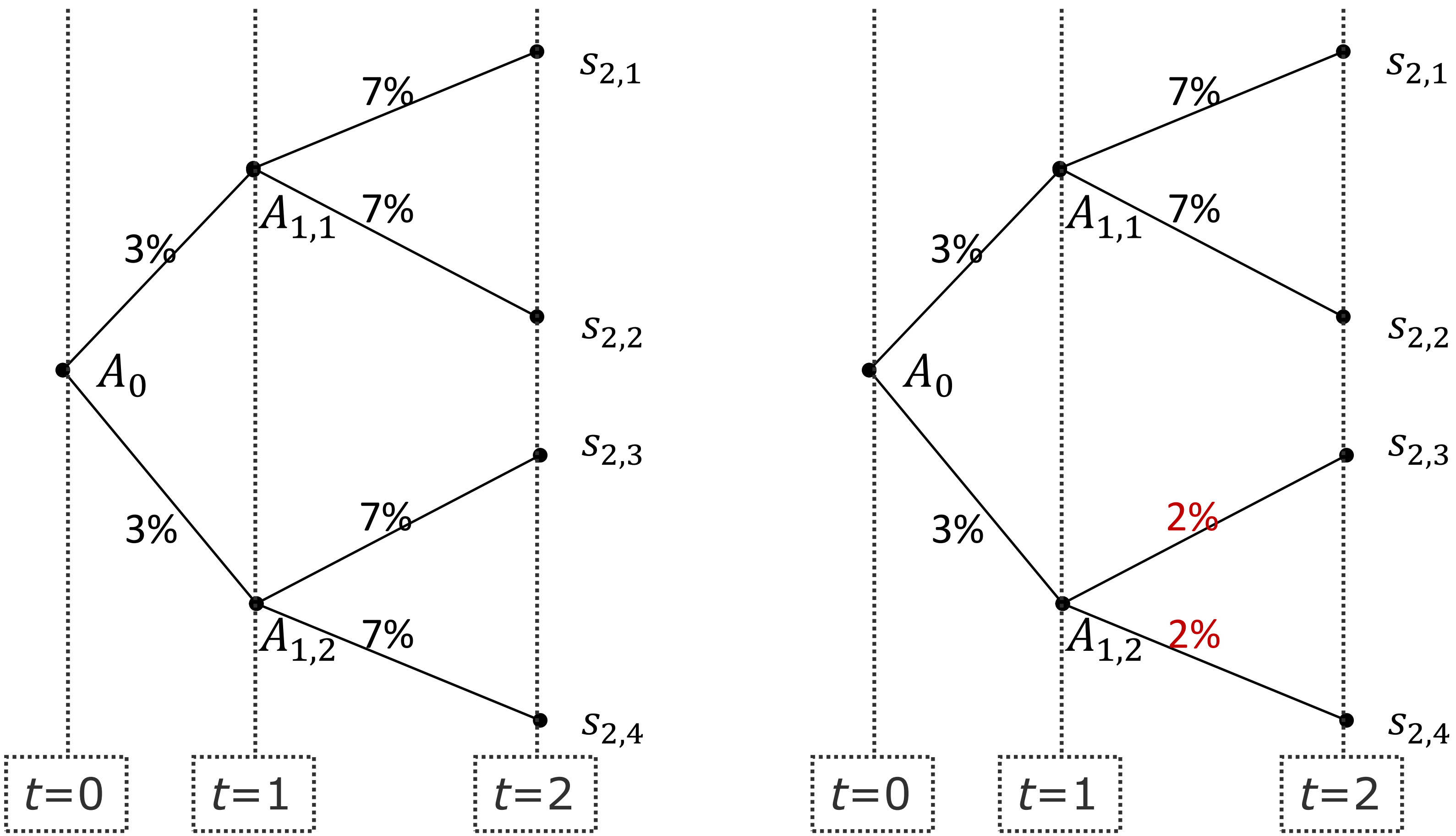
- One year zero-coupon bond: $Z_0(0,1) = 0.943396$
  - Pay $0.943396 today to receive $1 at t=1
  - Yield to maturity $YTM = \frac{1}{0.943396} - 1 = 0.06 = 6\% = r_0(0,1)$
- Two year zero-coupon bond: $Z_0(0,2) = 0.881659$
  - $YTM = \frac{1}{0.881659} - 1 = 0.134225 = (1 + r_0(0,2))^2 \Rightarrow r_0(0,2) = 0.065 = 6.5\%$

Yield Curve and Forward Curve

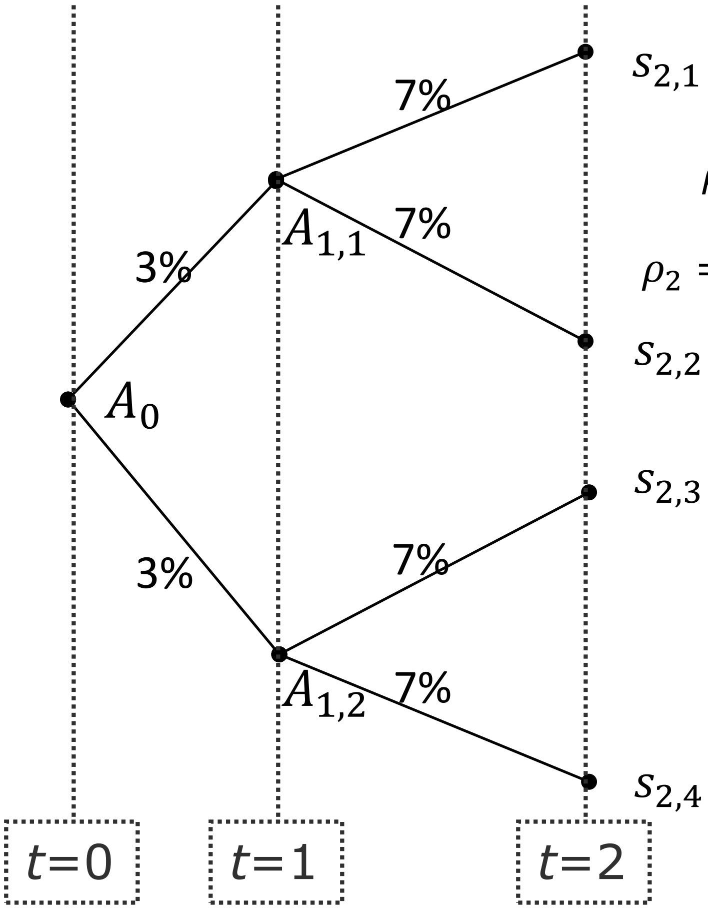


- Connection between
  - yield and forward curve
  - Forward rates and forward contracts discussed earlier

Deterministic vs. Stochastic Rate

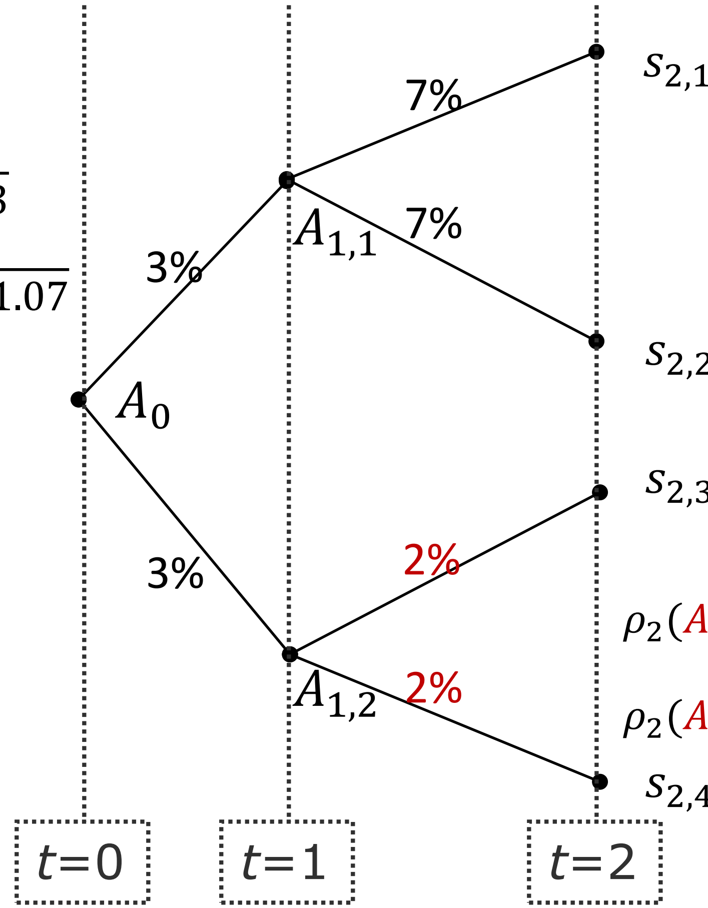


Deterministic vs. Stochastic Rate



$$\rho_1 = \frac{1}{1.03}$$

$$\rho_2 = \frac{1}{1.03 * 1.07}$$



$$\rho_1 = \frac{1}{1.03}$$

$$\rho_2(A_{1,1}) = \frac{1}{1.03 * 1.07}$$

$$\rho_2(A_{1,2}) = \frac{1}{1.03 * 1.02}$$

Log Interest Rate

Note: here $r_t(t, t+1) = r_t$

- Define $R_t =: e^{r_t}$
  - Compounding: $\prod_t^T R_\tau = \prod_t^T e^{r_\tau} = e^{\sum_t^T r_\tau}$
  - Discounting: $\prod_t^T \frac{1}{R_\tau} = \prod_t^T e^{-r_\tau} = e^{-\sum_t^T r_\tau}$
- In continuous time: $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$
- Approximate
  - $e^{r_t} \approx e^0 + e^0 r_t + HOT = 1 + r_t + HOT$
- With uncertainty
  - $E[R_t] = E[e^{r_t}] \neq e^{E[r_t]}$, $E[{}^1/_{R_t}] \neq {}^1/_{E[R_t]}$
  - With $R_t \sim \mathcal{N}$, r_t log-normal $E[e^{r_t}] = e^{E[r_t] + \frac{1}{2}Var[r_t]}$
- Bond yield: $e^{-y_t^{(N)} N} = Z_t(t, t+N) \Leftrightarrow \; y_t^{(N)} = -\frac{1}{N}\log Z_t(t, t+N)$

Coupon Bonds

- Price at time of issue of t of a bond maturing at time T that pays T fixed coupons of size c and maturity payment of \$1:

$$B_t(t,T) = \sum_{\tau=1}^{T} c Z_t(t,\tau) + Z_t(t,T)$$

- to sell at par, i.e. $B_t(t,T) = 1$ (face value) the coupon size must be:

$$c = \frac{1 - Z_t(t,T)}{\sum_{\tau=1}^{T} Z_t(t,\tau)}$$

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Duration

1. sensitivity of a bond's price to changes in interest rates
2. Average time it takes to get money back (roughly)

- Duration Measures:
  - Duration: $ change in price for a unit change in yield

$$-\frac{\Delta B(y)}{\Delta y} = \frac{1}{1+y}\sum_{\tau=1}^{T} \tau \frac{X_\tau}{(1+y)^\tau}$$

divide by 100 (10,000) for change in price given a 1% (1 basis point) change in yield

  - Macaulay Duration: (% change in price for % change in YTM, elasticity)

$$-\frac{\Delta B(y)\,/B(y)}{\Delta y/(1+y)} = \frac{1}{B(y)}\sum_{\tau=1}^{T} \tau \frac{X_\tau}{(1+y)^\tau}$$

  - y: yield per period; to annualize divide by the number of payments per year
  - $B(y)$: bond price as a function of yield y
  - X_τ payoff at time τ (coupon or principal)

Duration

- Example
  - 3-year zero-coupon bond with maturity value of \$100
    - Bond price at YTM of 7.00%: \$100/(1.0700^3)=\$81.62979
    - Bond price at YTM of 7.01%: \$100/(1.0701^3)=\$81.60691
    - Δ=-\$0.02288
    - Duration: $-\frac{1}{1.07} \times 3 \times \frac{\$100}{1.07^3} = -\$228.87$
    - For a basis point (0.01%) change: -\$228.87/10,000=-\$0.02289
    - Macaulay duration: $-(-\$228.87) \times \frac{1.07}{\$81.62979} = 3.000$
- Example
  - 3-year annual coupon (6.95485%) par bond
    - Macaulay Duration:

$$\left(1 \times \frac{0.0695485}{1.0695485}\right) + \left(2 \times \frac{0.0695485}{1.0695485^2}\right) + \left(3 \times \frac{1.0695485}{1.0695485^3}\right) = 2.80915$$

Duration Matching

- Match 1 bond with time to maturity t_1, price B_1, and Macaulay duration D_1 with
- N of different bond with time to maturity t_2, price B_2, Macaulay duration D_2
- Such that value of the resulting portfolio with duration zero is $B_1 + NB_2$

$$-\frac{\frac{\Delta B_1(y_1)}{B_1(y_1)}}{\frac{\Delta y_1}{1+y_1}} B_1(y_1)/(1+y_1) = -N\frac{\frac{\Delta B_2(y_2)}{B_2(y_2)}}{\frac{\Delta y_2}{1+y_2}} B_2(y_2)/(1+y_2)$$

$$N = -\frac{D_1 B_1(y_1)}{D_2 B_2(y_2)} \frac{1+y_2}{1+y_1}$$

- **Caveats**:
  - Duration is only a **first order** (linear) Taylor approximation
  - Duration matching only works for **parallel shifts** of the yield curve

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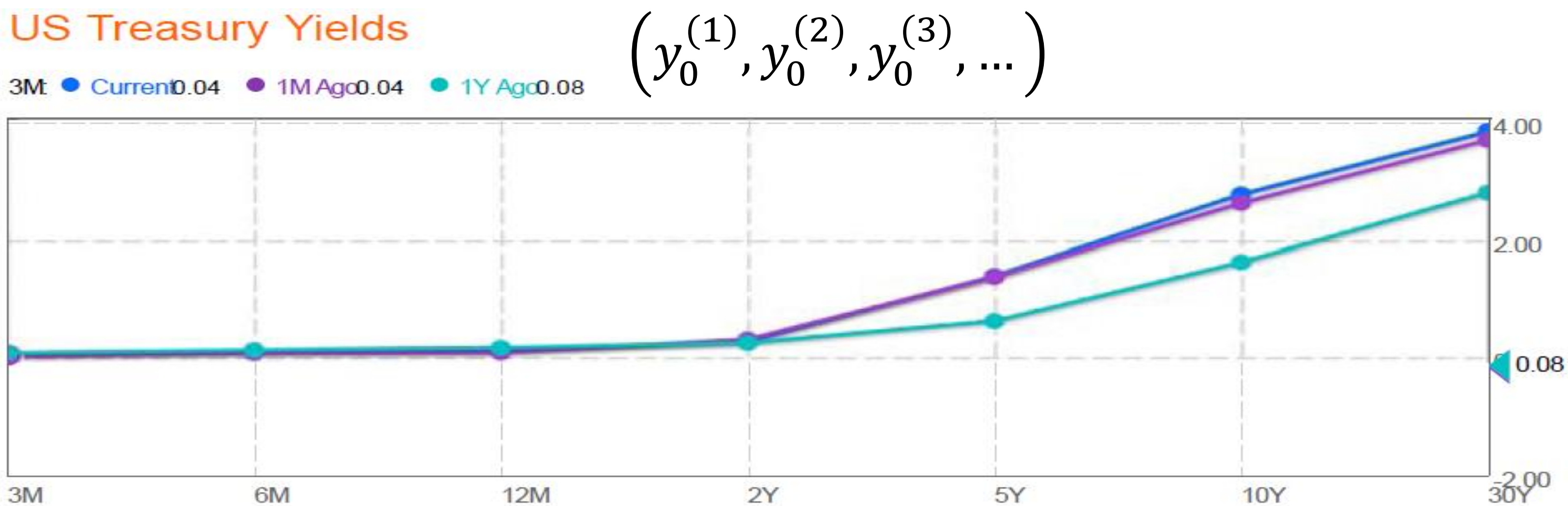
Cross-Section vs. Time-series View

- *cross section of prices*:
  The term structure are bond prices *at a particular point in time*. This is a cross section of prices.
- *time series properties*:
  how do interest rates evolve as time goes by?
- Time series view is the relevant view for an investor how tries to decide what kind of bonds to invest into, or what kind of loan to take.

Term Structure of Interest

$$\left(y_0^{(1)}, y_0^{(2)}, y_0^{(3)}, \dots\right)$$



Term Structure of Interest Rates

- Nominal versus real yield curve
- Three principal components (Litterman-Scheinkman 1991)
  - Level
  - Slope "term spread"
  - Curvature
- Long-end and slope of yield curve
  - Expectations about future short rate
    - Real: Expectations about future economic growth
    - Nominal: Expectations about future inflation
  - Risk premium
    - Real: Rollover risk
    - Nominal: Inflation risk

Real Term Structure & Economic Growth

- Risk-free zero coupon bond

$$Z_0(0,t) = E[M_t] = \frac{1}{\left(1 + y_0^{(t)}\right)^t}$$

- The corresponding (gross) yield is

$$1 + y_0^{(t)} = (Z_0(0,t))^{-\frac{1}{t}} = \delta^{-1} \left(\frac{E[u'(c_t)]}{u'(c_0)}\right)^{-\frac{1}{t}}$$

Since $m_{t+1} = \delta \frac{u'(c_{t+1})}{u'(c_t)}$, assuming representative agent with utility $E[\sum_t \delta^t u(c_t)]$

Real Term Structure & Economic Growth

$$1 + y_0^{(t)} = \delta^{-1} \left(\frac{E[u'(c_t)]}{u'(c_0)} \right)^{-\frac{1}{t}}$$

- Let g_t (state dependent) growth rate per period, so
$$(1 + g_t)^t = \frac{c_t}{c_0}.$$
- If representative agent with CRRA utility $\gamma = RRA$ first-order Taylor approximations yields
$$y_0^{(t)} \approx \gamma E[g_t] - \log[\delta]$$
Homework!
- (real) yield curve measures expected growth rates over different horizons.

Real Term Structure & Economic Growth

$$y_0^{(t)} \approx \gamma E[g_t] - \log[\delta]$$

- Second order Taylor approximation would include u'''-terms
  - Now, uncertainty about g_t also matters
  - If representative agent is prudent then uncertainty about g_t lowers yield.

- When is real term structure upward sloping?
  - Expected growth rate increases over time
  - long horizon uncertainty about the per capita growth rate is smaller than about short horizons
    (for instance if growth rates are mean reverting)

Term Structure & Risk Premium

- Need to invest for 2 periods
- Three possible options
  1. Buy 2-period ZC bond, yielding a (per period) return rate of $y_0^{(2)}$.
  2. Buy 1-period ZC bond and roll over when it matures.
     Expected yield: $\left(1+y_0^{(1)}\right)E_0[1+r_1(1{,}2)]$
     - Risky since period 1 spot rate is not known at $t=0$. (Rollover Risk)
  3. Buy 3-period ZC bond and sell after 2 periods.
     - Risky since price of 3-period ZC bond at $t=2$ is not known at $t=0$.
- Additional risk
  - Investor might know his investment horizon at $t=0$.
    - Since he faces random liquidity needs/endowment shocks.
  - Might want to hold liquid/safe asset.
  - Liquidity problem, (might mean revert with time horizon)

Expectations Hypothesis

- Pure expectations hypothesis
  - Term structure is purely determined by expectations about future short-term interest rate
  - No risk premia
- Expectations hypothesis (more generally)
  - Risk premia that are maturity dependent, but *constant through time*

Expectations Hypothesis (3 ways)

- Forward-rate view
  - Forward rate at t from $t+N \rightarrow t+N+1$ equal the expected future spot rate
  - $r_t(t+N, t+N+1) = E_t[y_{t+N}^{(1)}]$ (+ risk premium$^{(N)}$)
- Short-term view
  - Single-period holding returns on all maturity bonds are equal in expectations
  - $E_t\left[ln \frac{Z_{t+1}^{(N)}}{Z_t^{(N)}}\right] = E_t\left[ln \frac{e^{-y_{t+1}^{(N)}(N-1)}}{e^{-y_t^{(N)}N}}\right] = Ny_t^{(N)} - (N-1)E_t[y_{t+1}^{(N-1)}] = y_t^{(1)}$

    Note: here $Z_t^{(N)} = Z_t(t, t+N)$
- Long-term view
  - Multi-period holding returns on bonds of all maturities are the same in expectation
  - $y_t^{(N)} = \frac{1}{N} E_t[y_t^{(1)} + y_{t+1}^{(1)} + \cdots + y_{t+N-1}^{(1)}]$ (+ risk premium$^{(N)}$)

Empirical Evidence on EH: Long-term View

- $y_t^{(N)} - y_t^{(1)} = \frac{1}{N} E_t[\sum_{j=0}^{N-1} (y_{t+j}^{(1)} - y_t^{(1)})]$
- Yield spread forecasts long-term changes in yields on short-term bonds

Table 1
Means and Standard Deviations of Term Structure Variables

	Long bond maturity (months)						
Variable	*2*	*3*	*6*	*12*	*24*	*48*	*120*
Excess return	0.379 (0.640)	0.553 (1.219)	0.829 (2.950)	0.862 (6.203)	0.621 (11.29)	0.475 (19.32)	−0.234 (36.77)
Change in yield	0.014 (0.591)	0.014 (0.575)	0.014 (0.569)	0.014 (0.546)	0.014 (0.486)	0.014 (0.408)	0.013 (0.307)
Yield spread	0.196 (0.210)	0.324 (0.301)	0.569 (0.437)	0.761 (0.594)	0.948 (0.799)	1.141 (1.013)	1.358 (1.234)

Source: Author's calculations using estimated monthly zero-coupon yields, 1952–1991, from McCulloch and Kwon (1993). The data are measured monthly, but expressed in annualized percentage points. Each row shows the mean of the variable, with the standard deviation below in parentheses. Excess returns and yield spreads are measured relative to 1-month Treasury bill rates.

Empirical Evidence on EH: Short-term view

- $y_t^{(N)} - y_t^{(1)} = (N-1)E_t[y_{t+1}^{(N-1)} - y_t^{(N)}]$
- Yield spread forecasts short-term changes in yields on long-term bond.

Table 2

Regression Coefficients

	Long bond maturity (months)						
Dependent variable	*2*	*3*	*6*	*12*	*24*	*48*	*120*
Short-run changes in long yields	0.019 (0.194)	−0.135 (0.285)	−0.842 (0.444)	−1.443 (0.598)	−1.432 (0.996)	−2.222 (1.451)	−4.102 (2.083)
Long-run changes in short yields	0.510 (0.097)	0.473 (0.149)	0.301 (0.147)	0.253 (0.210)	0.341 (0.221)	0.435 (0.398)	1.311 (0.120)

Empirical Evidence on EH

- When the yield spread is unusually high
  - Long-term view
    short-term interest rates do tend to rise,
    but not as much as predicted by EH.
  - Short-term view
    yield on the long-term bonds tends to fall,
    not rise as predicted by EH.

- Term structure models with time-varying risk premia needed.

Violation of EH due to $E[e^r] \neq e^{E[r]}$

- Strictly speaking, the PEH in log-rates does not hold precisely even when agents are risk neutral
  - $Z_t(t, t+N) = e^{-y_t^{(N)} N} = E_t[e^{-\sum_0^N r_t(t,t+\tau)}]$
  - when r_t stochastic

$$y_t^{(N)} N \neq E_t[\sum_0^N r_t(t, t+\tau)]$$

    - since $E[e^r] \neq e^{E[r]}$
    - E.g. if r is normal, then $E[e^r] = e^{E[r] + \frac{1}{2} Var[r]}$
- Discount factor EH doesn't suffer from this.

Pure Expectations Hypothesis in Discount Factor

- Consider a t-period zero coupon bond. The price is
$$Z_0(0,t) = E[M_t] = E[m_1 \cdots m_t]$$
  - Invest $Z_0(0,t)$in $t = 0$, receive one consumption unit in period t.

- Alternatively, buy 1-period discount bonds and roll them over t-times. The investment that is necessary today to get one consumption unit (in expectation) in period t
$$E[m_1] \cdots E[m_t]$$

(to see this for $t = 2$: buying at $t = 0$ $E[Z_1(1,2)]$ bonds with maturity $t = 1$ costs $Z_0(0,1)E[Z_1(1,2)]$ and pays $E[Z_1(1,2)]$at $t = 1$, which allows –in expectation- to pay for a bond with maturity $t = 2$ which finally pays $1 at $t = 2$)

Pure Expectations Hypothesis in terms of Discount Factor

- Two strategies yield same expected return rate if and only if
$$E[m_1 \cdots m_t] = E[m_1] \cdots E[m_t]$$
which holds if m_t is serially uncorrelated.

  - Special examples:
    - World of certainty
    - Risk-neutral world
    - i.i.d world
  - In that case, no term premia
    assumption known as the expectations hypothesis.
  - Whenever m_t is serially correlated (for instance because the growth process is serially correlated), then expectations hypothesis may fail.

Expectations Hypothesis

- Homework:
  1. Show the equivalence of the three ways to present the expectations hypothesis.
  2. Show whether under the pure expectations hypothesis in terms of discount factor the risk-neutral measure coincides with the risk forward measure.

Term Structure Models

Beyond Expectations Hypothesis

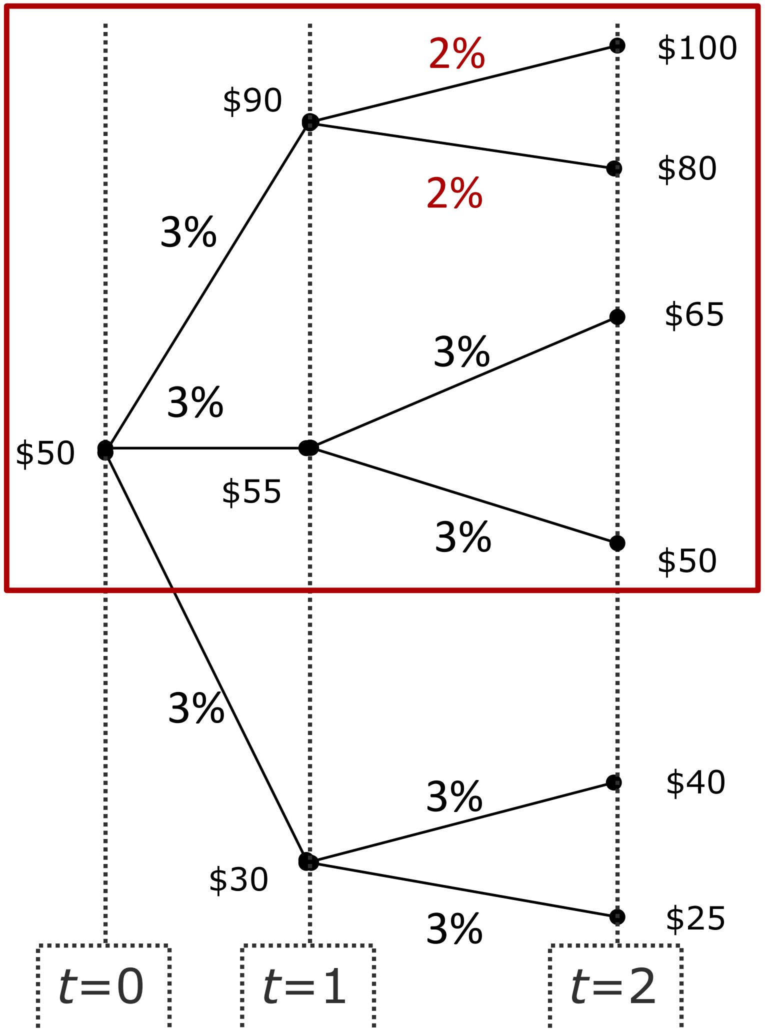
- Specify process for
  - SDM M_t^* for
    - Time-varying risk premium
    - for short-rate in P-measure
- Specify process for
  - for short-rate in Q-measure
- Canonical models
  - Vasicek
  - CIR
  - Affine

Canonical Term Structure Models

Term Structure Model (under Q)	**Features**
Vasicek: $$r_{t+1} = r_t + k(\theta - r_t) + \sigma\varepsilon_{t+1}$$	Very easy to use (AR model), rates can be negative, constant volatility
Cox-Ingersoll-Ross: $$r_{t+1} = r_t + k(\theta - r_t) + \sigma\sqrt{r_t}\varepsilon_{t+1}$$	Rates cannot be negative, volatility is high when rates are high (empirical fact)
Affine Term Structure (example): $$\theta_{t+1} = \theta_t + v(\bar{\theta} - \theta_t) + \gamma\sqrt{\theta_t}\varepsilon_{t+1}^2$$ $$u_{t+1} = u_t + \mu(\bar{u} - u_t) + \delta\sqrt{u_t}\varepsilon_{t+1}^3$$	Multi-factor model: better calibration than the others, harder to handle. Give rise to ZCB price of the type $Z_t(t,T) = e^{a(T-t)+\sum_i b_i(T-t)r_t}$ Param. values ensure existence

Note: here $r_t(t, t+1) = r_t$

Interest Rates, Stocks and State Space



- If we consider at the same time a stock and interest rates, we have multiple sources of uncertainty, perhaps correlated. To account for this we need to expand the state space to include all possible combination of stock-interest rates pairs.
- If we are only interested in interest rates we can just collapse the tree to the sub-tree in the red box, and the new state space will capture all the information we need.

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Repurchase Agreements

- A repurchase agreement or a repo entails selling a security with an agreement to buy it back at a fixed price
  - Sale + forward to repurchase
- The underlying security is held as collateral by the counterparty
  $\Rightarrow$ A repo is collateralized borrowing
- Used by securities dealers to finance inventory
- A "haircut" is charged by the counterparty to account for credit risk

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Literature

- C. Plueger and Luis Viceira, 2011, “Inflation-Indexed Bonds and the Expectations Hypothesis”, Annual Review of Financial Economics
- David Backus, Silverio Foresi and Chris Telmer, Discrete-Time Models of Bond Pricing (CMU website)