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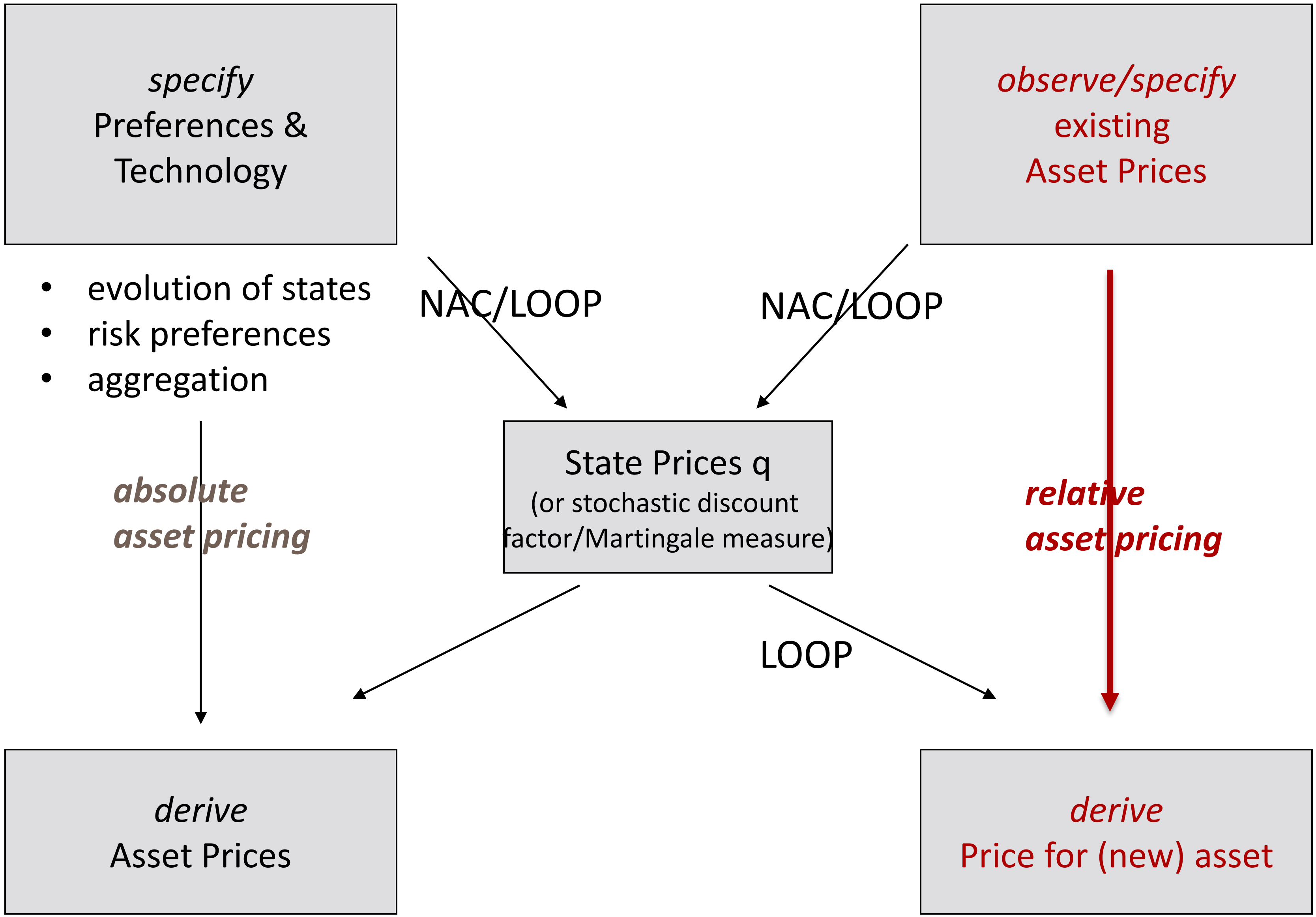
LECTURE 11: MULTI-PERIOD EQUILIBRIUM MODEL



Only works as long as market completeness doesn't change

Time-varying R_t^* (SDF)

- If one-period SDF m_t is not time-varying (i.e. distribution of m_t is i.i.d., then
  - Expectations hypothesis holds
  - Investment opportunity set does not vary
  - Corresponding R* of **single factor** state-price beta model can be easily estimate (because over time one more and more observations about R*)
- If not, then m_t (or corresponding R_t^*)
  - depends on state variable
  - **multiple factor** model

R_t^* depends on State Variable

- $R_t^* = R^*(z_t)$, with state variable z_t
- Example:
  - $z_t = 1$ or 2 with equal probability
  - Idea:
    - Take all periods with $z_t = 1$ and figure out $R^*(1)$
    - Take all periods with $z_t = 2$ and figure out $R^*(2)$
  - Can one do that?
    - No – hedge across state variables
- Potential state-variables: predict future return

Dynamic Hedging Demand

- Trade-off
  - Low return realization in next period
    - Good since opportunity going forward is high
      - Invest more
    - Bad since marginal utility is high
      - Consume and invest less
  - High return realization in next period ....
- Utility
  - $\gamma > (<) 1$ first (second) effect dominates
  - $\gamma = 1$ (log-utility) both effects offset each other

(Dynamic) Hedging Demand

- Illustration with noise trader risk:
  - Suppose fundamental value is constant v=1, but price is noisy (due to noise traders)
  - If the asset is underpriced, e.g. p=.9 , then it might be even more underpriced in the next period
    - Myopic risk-averse investor:
      buy some of the asset and push price towards 1, but not fully
    - Forward-looking risk-averse investor:
      yes, there can be intermediate losses if price declines in next period, but then **investment opportunity set** improves even more i.e. if returns are bad, then I have great opportunity (dynamic hedge)

Static problem = intertemporal problem

- In general ICAPM setting
  - CRRA with $\gamma \neq 1$ and changing investment opportunity sets
- Special cases
  1. CRRA and i.i.d. returns and constant r^f
     - SR and LR investors have the same portfolio weights.
     - Fraction of savings that is invested in asset j is time-invariant (Merton 1971)
  2. Log utility and non-i.i.d. returns $\Rightarrow$ same result

Conditional vs. unconditional CAPM

- If β of each subperiod CAPM are time-independent, then
  conditional CAPM = unconditional CAPM
- If βs are time-varying they may co-vary with R_m and hence CAPM equation does not hold for unconditional expectations.
  - Additional co-variance terms have to be considered!
  - Move from single-factor setting to multi-factor setting

Intertemporal CAPM (ICAPM)

- Merton (1973)
- Bellman equation

$$V(W_t, z_t) = \max_{c}\{u(c_t) + \delta E_t[V(W_{t+1}, z_{t+1})\}$$

  – where $W_{t+1} = R_{t+1}^W (W_t - c_t)$ with R_{t+1}^W for optimal portfolio
- FOC:
  – $0 = u'(c_t) - \delta E_t[V_W(W_{t+1}, z_{t+1}) R_{t+1}^W]$
  – Since $V_W(W_t, z_t) = \delta E_t[V_W(W_{t+1}, z_{t+1}) R_{t+1}^W]$, (envelope theorem)

$$u'(c_t) = V_W(W_t, z_t)\ \forall t$$

Deriving ICAPM

- Hence one period pricing equation

$$E\left[R_{t+1}^{j}\right]-R_{t+1}^{f}=-Cov_t[\frac{u'(c_{t+1})}{E[u'(c_{t+1})]},R_{t+1}^{j}]$$

- Becomes

$$E\left[R_{t+1}^{j}\right]-R_{t+1}^{f}$$
$$=-\frac{Cov_t\left[V_W(W_{t+1},z_{t+1}),R_{t+1}^{j}\right]}{E_t[V_W(W_{t+1},z_{t+1})]}$$

Deriving ICAPM: First order Approximation

- Around $V_W(W_t, z_t)$
- $V_W(W_{t+1}, z_{t+1})$
  $\approx V_W(W_t, z_t) + V_{WW}(W_t, z_t)\Delta \mathrm{W}_{\mathrm{t+1}} + V_{Wz}(W_t, z_t)\Delta \mathrm{z}_{\mathrm{t}}$
- One obtains

$$E\left[R_{t+1}^j\right] - R_{t+1}^f$$

$$= -\gamma Cov_t\left[\Delta W_{t+1}, R_{t+1}^j\right] + \frac{V_{Wz}}{E_t[V_W]} Cov_t\left[\Delta z_{t+1}, R_{t+1}^j\right]$$

  - γ is relative risk aversion coefficient of V
  - Second term is additional “risk factor”

Approximate ICAPM Campbell

- CRRA Agent

$$\Rightarrow 1 = E_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{j,t+1}\right]$$

- 2nd order Taylor Approximation, where $V_{cc} \equiv \mathrm{var}_t[\Delta c_{t+1}] = \mathrm{var}[\Delta c_{t+1} - E_t \Delta c_{t+1}]$ etc.

$$0 = \log\delta - \gamma E_t \Delta c_{t+1} + E_t r_{j,t+1} + \frac{1}{2}\left[\gamma^2 V_{cc} + V_{jj} - 2\gamma V_{cj}\right]$$

- Note that this implies $E_t[\Delta c_{t+1}] = \mu_m + \frac{1}{\gamma} E_t\left[r_{m,t+1}\right]$
  - With $\mu_m = \frac{1}{\gamma}\log[\delta] + \frac{1}{2}\left[\gamma V_{cc} + \frac{1}{\gamma V_{mm}} - 2V_{cm}\right]$
  - $= \frac{1}{\gamma}\log[\delta] + \frac{1}{2}\gamma \mathrm{var}_t\left[\Delta c_{t+1} - \sigma r_{m,t+1}\right]$

Consumption

- Budget constraint

$$W_{t+1} = R_{m,t+1}(W_t - C_t) \Rightarrow \frac{W_{t+1}}{W_t} = R_{m-t+1}\left(1 - \frac{C_t}{W_t}\right)$$

- In logs

$$\Delta w_{t+1} = r_{m,t+1} + \log[1 - \exp(c_t - w_t)]$$

- 1st Order Taylor Approximation (1)

$$\log[1 - \exp[x_t]] \approx \log[1 - \exp[\bar{x}]] - \frac{\exp[\bar{x}]}{1 - \exp[\bar{x}]}(x_t - \bar{x})$$

$$\Delta w_{t+1} \approx r_{m,t+1} + k + (1 - \rho^{-1})(c_t - w_t)$$

Where $\rho \equiv 1 - \exp(\bar{c} - \bar{w})$

Consumption Innovations

- Since $\Delta w_{t+1} = \Delta c_{t+1} + (c_t - w_t) - (c_{t+1} - w_{t+1})$ (Simply an identity)

$$\Rightarrow c_t - w_t = \sum_{\tau=1}^{\infty} \rho^{\tau}\left(r_{m,\tau+j} - \Delta c_{t+\tau}\right) + \frac{\rho k}{1-\rho}$$

- Taking expectations on both sides yields (2)

$$c_t - w_t = E_t\left[\sum_{\tau=1}^{\infty} \rho^{\tau}\left(r_{m,\tau+j} - \Delta c_{t+\tau}\right) + \frac{\rho k}{1-\rho}\right]$$

- Combining (1) and (2)

$$c_{t+1} - E_t c_{t+1}$$

$$= (E_{t+1} - E_t)\sum_{\tau=0}^{\infty} \rho^{\tau} r_{m,t+1+\tau} - (E_{t+1} - E_t)\sum_{\tau=1}^{\infty} \rho^{\tau} \Delta c_{t+1+\tau}$$

Consumption Innovations

- Combining previous result with $E_t[\Delta c_{t+1}] = \mu_m + \frac{1}{\gamma} E_t\left[r_{m,t+1}\right]$

$$\Rightarrow c_{t+1} - E_t c_{t+1}$$

$$= r_{m,t+1} - E_t r_{m,t+1} + \left(1 - \frac{1}{\gamma}\right) (E_{t+1} - E_t) \sum_{\tau=0}^{\infty} \rho^{\tau} r_{m,t+1+\tau}$$

- Finally, this implies that $V_{jc} = V_{jm} + \left(1 - \frac{1}{\gamma}\right) V_{jh}$
  - Here, we define $V_{ih} = \mathrm{cov}_t\left[r_{j,t+1}, (E_{t+1} - E_t) \sum_{\tau=0}^{\infty} \rho^{\tau} r_{m,t+1+\tau}\right]$
  - This is the covariance of the asset with a “hedge” portfolio

ICAPM

- For a risk-free asset the log-Euler equation simplifies to

$$0 = \log \delta - \gamma E_t \Delta c_{t+1} + r_{f,t+1} + \frac{1}{2}\gamma^2 V_{cc}$$

- Then we can write the Consumption CAPM

$$E_t r_{j,t+1} - r_{f,t+1} = -\frac{V_{jj}}{2} + \gamma V_{jc}$$

- And finally the Intertemporal CAPM

$$E_t r_{j,t+1} - r_{f,t+1} = -\frac{V_{jj}}{2} + \gamma V_{jm} + (\gamma - 1) V_{jh}$$