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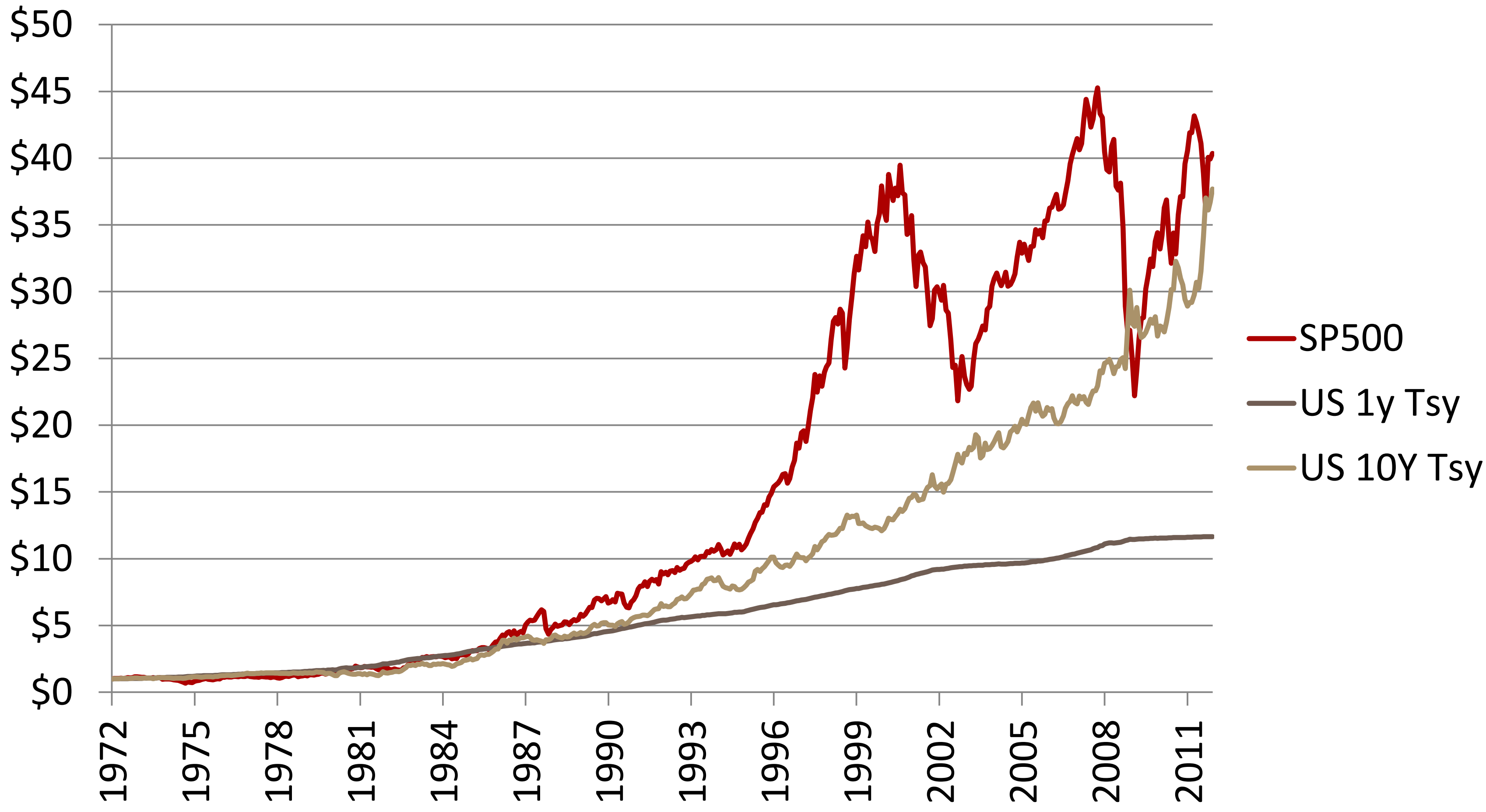
LECTURE 06: SHARPE RATIO, BONDS, & THE EQUITY PREMIUM PUZZLE

Money, Bonds vs. Stocks



Sharpe Ratios and Bounds

- Consider a one period security available at date t with payoff x_{t+1}. We have
$$p_t = E_t[m_{t+1}x_{t+1}]$$
or
$$p_t = E_t[m_{t+1}]E_t[x_{t+1}] + \text{cov}[m_{t+1}, x_{t+1}]$$

- For a given m_{t+1} we let $R^f_{t+1} = \frac{1}{E_t[m_{t+1}]}$
  - Note that R^f_{t+1} will depend on the choice of m_{t+1} unless there exists a riskless portfolio
  - R_{t+1} is the return from t to $t+1$, typically measurable w.r.t. $\mathcal{F}_{t+1}$. (An exception is R^f, which is measurable w.r.t. $\mathcal{F}_t$, but we stick with subscript $t+1$.)

Sharpe Ratios and Bounds (ctd.)

- Hence

$$p_t = \underbrace{\frac{1}{R^f_{t+1}} E_t[x_{t+1}]}_{\text{Expected PV}} + \underbrace{\text{cov}[m_{t+1}, x_{t+1}]}_{\text{Risk Adjustment}}$$

  – Positive correlation with the discount factor adds value, i.e. decreases required return

in Returns

$$E_t[m_{t+1}x_{t+1}] = p_t$$

– Divide both sides by p_t and note that $\frac{x_{t+1}}{p_t} = R_{t+1}$

$$E_t[m_{t+1}R_{t+1}] = 1$$

– Using $R^f_{t+1} = 1/E_t[m_{t+1}]$, we obtain

$$E_t\left[m_{t+1}\left(R_{t+1} - R^f_{t+1}\right)\right] = 0$$

– m-discounted expected excess return for all assets is zero.

in Returns

– Since $E_t[m_{t+1}(R_{t+1} - R^f_{t+1})] = 0$

$$\text{cov}_t[m_{t+1}, R_{t+1} - R^f_{t+1}]$$
$$= -E_t[m_{t+1}]E_t[R_{t+1} - R^f_{t+1}]$$

- That is, risk premium or expected excess return

$$E_t[R_{t+1} - R^f_{t+1}] = -\frac{\text{cov}_t[m_{t+1}, R_{t+1}]}{E_t[m_{t+1}]}$$

is determined by its covariance with the stochastic discount factor

Sharpe Ratio

- Multiply both sides with portfolio h

$$E_t\left[\left(R_{t+1} - R^f_{t+1}\right)h\right] = -\frac{\text{cov}_t[m_{t+1}, R_{t+1}h]}{E_t[m_{t+1}]}$$

$$E_t\left[\left(R_{t+1} - R^f_{t+1}\right)h\right] = -\frac{\rho(m_{t+1}, R_{t+1}h)\sigma(R_{t+1}h)\sigma(m_{t+1})}{E_t[m_{t+1}]}$$

  - NB: All results also hold for unconditional expectations $E[\cdot]$

- Rewritten in terms of Sharpe Ratio = ...

$$-\frac{\sigma(m_{t+1})}{E[m_{t+1}]}\rho(m_{t+1}, R_{t+1}h) = \frac{E\left[\left(R_{t+1} - R^f_{t+1}\right)h\right]}{\sigma(R_{t+1}h)}$$

Hansen-Jagannathan Bound

- Since $\rho \in [-1,1]$ we have

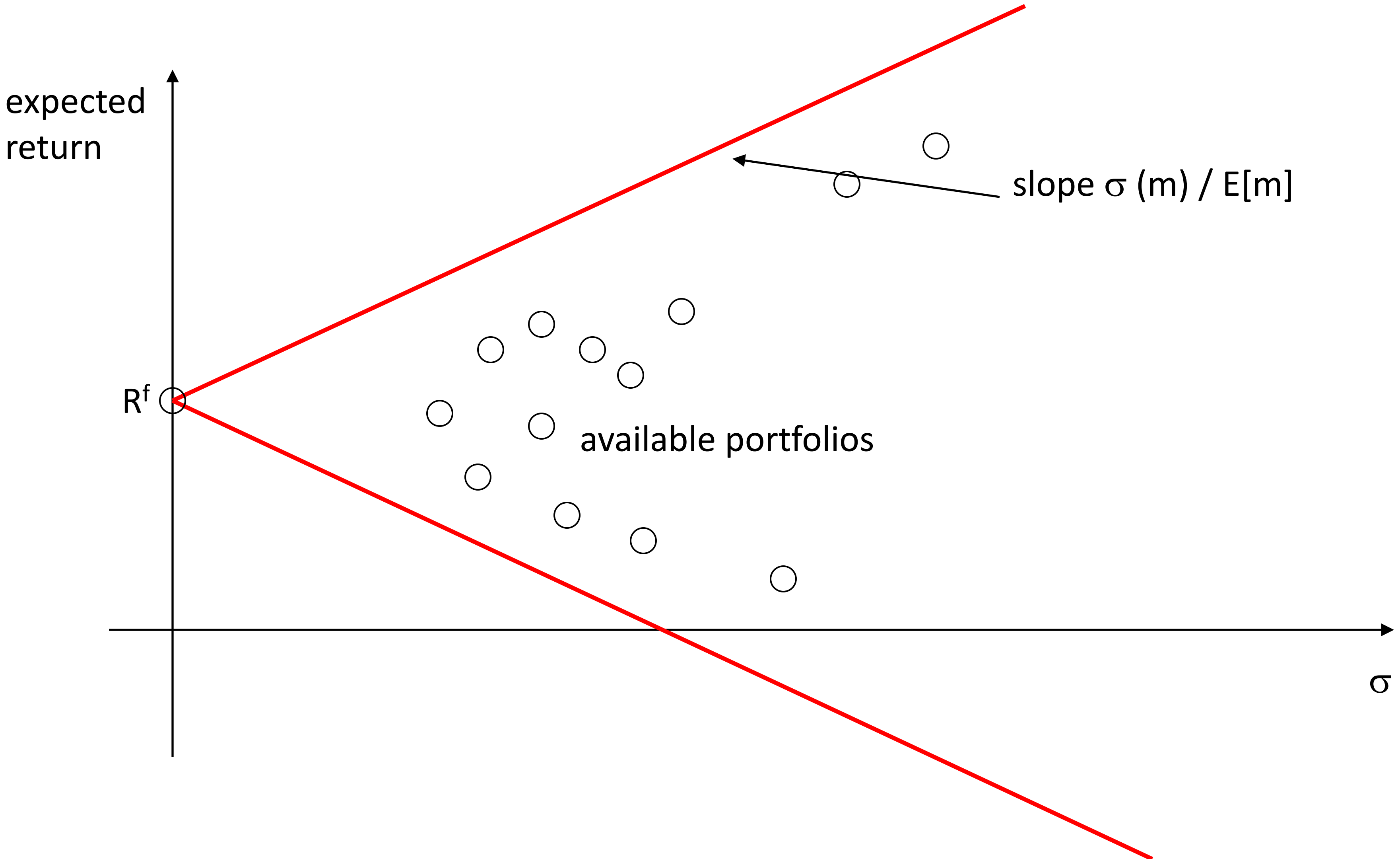
$$\frac{\sigma(m_{t+1})}{E[m_{t+1}]} \geq \sup_h \left| \frac{E\left[\left(R_{t+1} - R_{t+1}^f\right)h\right]}{\sigma(R_{t+1}h)} \right|$$

- **Theorem (Hansen-Jagannathan Bound):** The ratio of the standard deviation of a stochastic discount factor to its mean exceeds the Sharpe Ratio attained by any portfolio.

Hansen-Jagannathan Bound

- **Theorem (Hansen-Jagannathan Bound):** The ratio of the standard deviation of a stochastic discount factor to its mean exceeds the Sharpe Ratio attained by any portfolio.
  - Can be used to easy check the “viability” of a proposed discount factor
  - Given a discount factor, this inequality bounds the available risk-return possibilities
  - The result also holds conditional on date *t* info

Hansen-Jagannathan Bound



Assuming Expected Utility

- $c_0 \in \mathbb{R}, c_1 \in \mathbb{R}^S$
- $U(c_0, c_1) = \sum_s \pi_s u(c_0, c_{1,s})$U($c_0$,$c_1$)

$$\partial_0 u = \left(\frac{\partial u(c_0^*, c_{1,1}^*)}{\partial c_0}, \dots, \frac{\partial u(c_0^*, c_{1,S}^*)}{\partial c_0} \right)$$

$$\partial_1 u = \left(\frac{\partial u(c_0^*, c_{1,1}^*)}{\partial c_{1,1}}, \dots, \frac{\partial u(c_0^*, c_{1,S}^*)}{\partial c_{1,S}} \right)$$

- Stochastic discount factor

$$m = \frac{\text{MRS}}{\pi} = \frac{\partial_1 u}{E[\partial_0 u]} \in \mathbb{R}^S$$

Time-Separable

- *Digression:* if utility is in addition time-separable $u(c_0, c_1) = v(c_0) + v(c_1)$
- Then

$$\partial_0 u = \left(\frac{\partial v(c_0^*)}{\partial c_0}, \dots, \frac{\partial v(c_0^*)}{\partial c_0}\right)$$

$$\partial_1 u = \left(\frac{\partial v\left(c_{1,1}^*\right)}{\partial c_{1,1}}, \dots, \frac{\partial v\left(c_{1,S}^*\right)}{\partial c_{1,S}}\right)$$

- And

$$m_s = \frac{1}{\pi_s} \frac{\pi_s v'\left(c_{1,s}\right)}{v'(c_0)} = \frac{v'\left(c_{1,s}\right)}{v'(c_0)}$$

A simple example

- $S = 2, \pi_1 = \frac{1}{2}$
- 3 securities with $x^1 = (1,0), x^2 = (1,0), x^3 = (1,1)$
- Let $m = \left(\frac{1}{2}, 1\right), \sigma = \frac{1}{4} = \sqrt{\frac{1}{2}\left(\frac{1}{2} - \frac{3}{4}\right)^2 + \frac{1}{2}\left(1 - \frac{3}{4}\right)^2}$
- Hence, $p^1 = \frac{1}{4}, p^2 = \frac{1}{2} = p^3 = \frac{3}{4}$ and
- $R^1 = (4,0), R^2 = (0,2), R^3 = \left(\frac{4}{3}, \frac{4}{3}\right)$
- $E[R^1] = 2, E[R^2] = 1, E[R^3] = \frac{4}{3}$

Example: Where does SDF come from?

- "Representative agent" with
  - Endowment: 1 in date 0, $(2,1)$ in date 1
  - Utility $EU(c_0, c_1, c_2) = \sum_s \pi_s \left(\ln c_0 + \ln c_{1,s}\right)$
  - i.e. $u\left(c_0, c_{1,s}\right) = \ln c_0 + \ln c_{1,s}$ (additive) time separable u-function
- $m = \frac{\partial_1 u(1,2,1)}{E[\partial_0 u(1,2,1)]} = \left(\frac{c_0}{c_{1,1}}, \frac{c_0}{c_{1,2}}\right) = \left(\frac{1}{2}, \frac{1}{1}\right) = \left(\frac{1}{2}, 1\right)$
  since endowment=consumption
  - Low consumption states are high "m-states"
  - Risk-neutral probabilities combine true probabilities and marginal utilities.

Equity Premium Puzzle

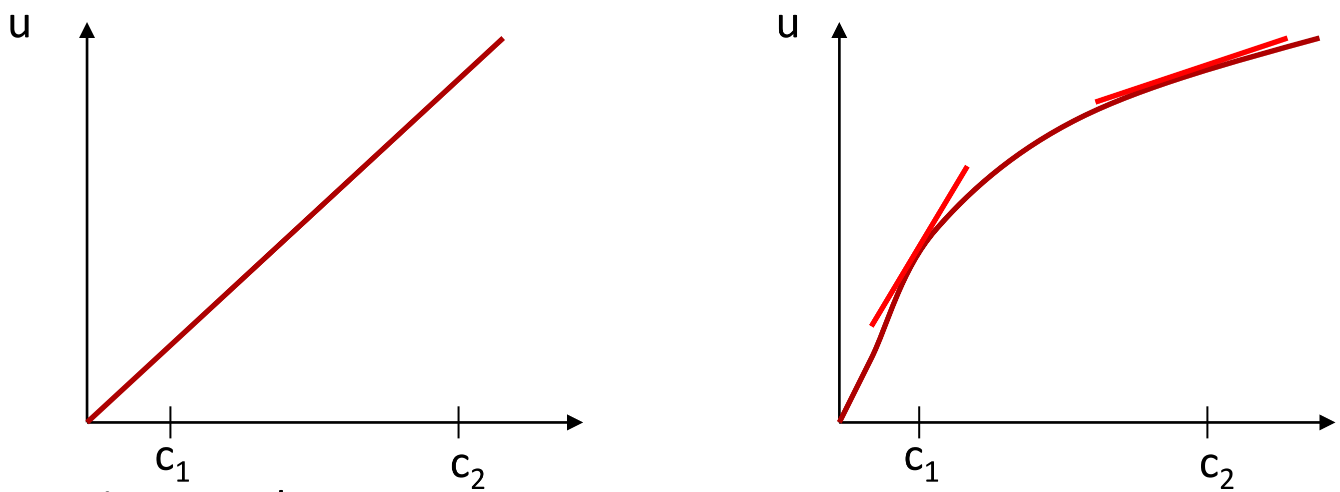
- Recall $E[R^i] - R^f = -R^f \operatorname{cov}[m, R^i]$
- Now: $E[R^j] - R^f = -\frac{R^f \operatorname{cov}[\partial_1 u, R^j]}{E[\partial_0 u]}$
- Recall Hansen-Jaganathan bound

$$\frac{\sigma(m)}{E[m]} \geq \left|\frac{E[R - R^f]}{\sigma(R)}\right|; E[m] = \frac{1}{R^f}$$

$$\sigma(m) \geq \frac{1}{R^f}\left|\frac{E[R - R^f]}{\sigma(R)}\right|$$

Equity Premium Puzzle (ctd.)

$$\sigma\left(\frac{\partial_1 u}{E[\partial_0 u]}\right) \geq \frac{1}{R^f}\left|\frac{E\left[R - R^f\right]}{\sigma(R)}\right|$$



Equity Premium Puzzle

- high observed Sharpe ratio of stock market indices
- low volatility of consumption
- $\Rightarrow$ (unrealistically) high level of risk aversion

Equity Premium or Low Risk-free Rate Puzzle?

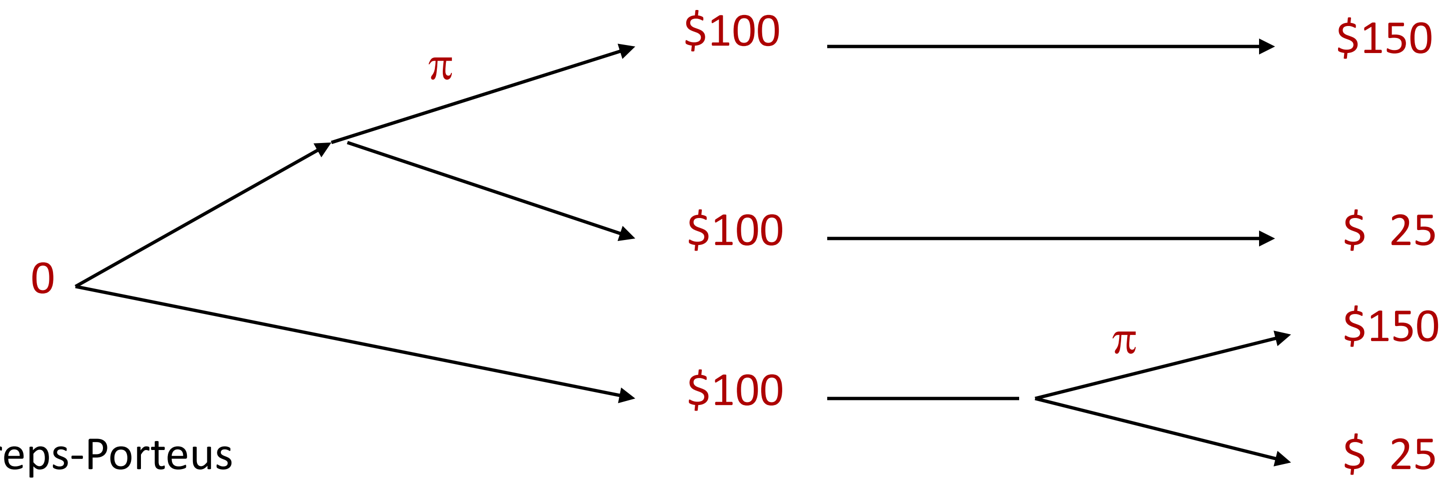
- Suppose we allow for sufficiently high risk aversion s.t. $\sigma\left(\frac{\partial_1 u}{E[\partial_0 u]}\right) \geq \frac{1}{R^f}\left|\frac{E\left[R-R^f\right]}{\sigma(R)}\right|$
- New problem emerges:
  - Strong force to consumption smooth over time (low intertemporal elasticity of consumption (IES)) due to concavity of utility function
  - vNM utility function
    - smoothing over states = smoothing over time
    - CRRA gamma = 1/ IES
  - Model predicts much higher risk-free rate

Equity Premium or Low Risk-free Rate Puzzle?

- Solution:
  - depart from vNM utility preference representation
  - Found preference representation that allows split
    - Risk-aversion
    - Intertemporal elasticity of substitution

- Kreps-Porteus (special case: vNM)
- Epstein-Zin (special case: CRRA vNM)

Digression: Preference for the timing of uncertainty resolution



Kreps-Porteus

$$U_0(x_1, x_2(s)) = W(x_1, E[U_1(x_1, x_2(s))])$$

Early (late) resolution if $W(P_1, \ldots)$ is convex (concave)

Do you want to know whether you will get cancer at the age of 55 now?