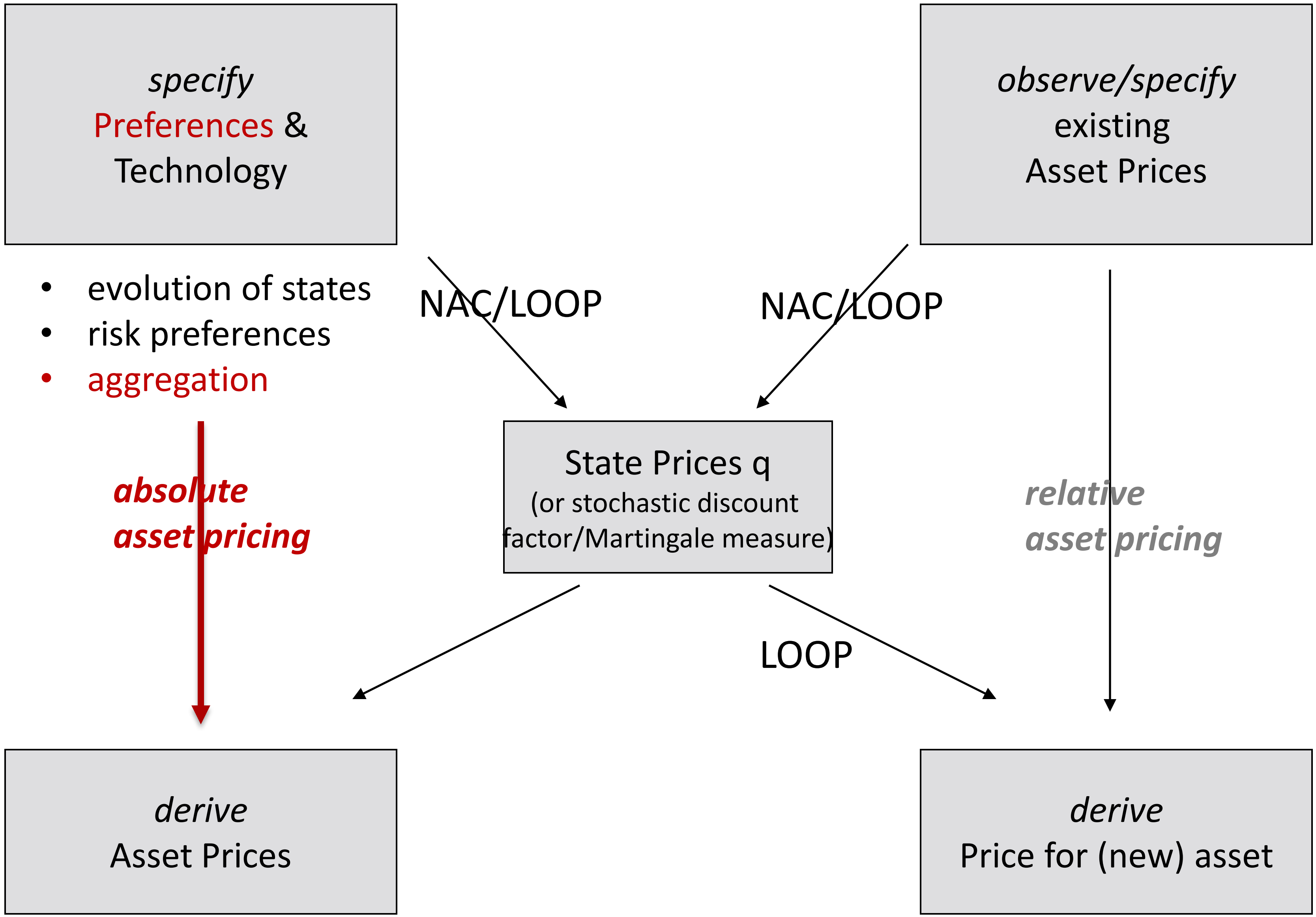
Markus K. Brunnermeier

LECTURE 05: ONE PERIOD MODEL GENERAL EQUILIBRIUM, EFFICIENCY & AGGREGATION



Only works as long as market
completeness doesn't change

Overview

1. Marginal Rate of Substitution (MRS)
2. Pareto Efficiency
3. Welfare Theorems
4. Representative Agent Economy

Representation of Preferences

A (i) complete, (ii) transitive, (iii) continuous [and (iv) relatively stable] preference ordering can be represented by a utility function, i.e.

$$(c_0, c_1, \dots, c_S) \succ (c'_0, c'_1, \dots, c'_S) \Leftrightarrow$$
$$U(c_0, c_1, \dots, c_S) > U(c'_0, c'_1, \dots, c'_S)$$

Agent's Optimization

- Consumption vector $(c_0, c_1) \in \mathbb{R}_+ \times \mathbb{R}_+^S$
- Agent i has $U^i: \mathbb{R}_+ \times \mathbb{R}_+^S \to \mathbb{R}$
  endowments $(e_0, e_1) \in \mathbb{R}_+ \times \mathbb{R}_+^S$
- U^i is quasiconcave $\{c: U^i(c) \geq v\}$ is convex for each real v

  U^i is concave: for each $0 \leq \alpha \leq 1$,
  $U^i(\alpha c + (1-\alpha)c') \geq \alpha U^i(c) + (1-\alpha)U^i(c')$
- $\frac{\partial U^i}{\partial c_0} > 0, \frac{\partial U^i}{\partial c_1} \gg 0$

Agent's Optimization

- Portfolio consumption problem

$$\max_{c_0, c_1, h} U^i(c_0, c_1)$$

subject to (i) $0 \leq c_0 \leq e_0 - p \cdot h$
and (ii) $0 \leq c_1 \leq e_1 + X'h$

$$\mathcal{L} = U^i(c_0, \vec{c}_1) - \lambda[c_0 - e_0 + ph] - \vec{\mu}[\vec{c}_1 - \vec{e}_1 - h'X]$$

- FOC

$$c_0 \colon \frac{\partial U^i}{\partial c_0}(c^*) = \lambda, \qquad c_s \colon \frac{\partial U^i}{\partial c_s}(c^*) = \mu_s$$

$$h \colon \lambda \vec{p} = X \vec{\mu}$$

$$\Leftrightarrow p^j = \sum_s \frac{\mu_s}{\lambda} x_s^j$$

Agent's Optimization

$$p^j = \sum_s \frac{\partial U^i / \partial c_s}{\partial U^i / \partial c_0} x_s^j$$

- For time separable utility function

$$U^i(c_0, \vec{c}_1) = u(c_0) + \delta u(\vec{c}_1)$$

- And vNM expected utility function

$$U^i(c_0, \vec{c}_1) = u(c_0) + \delta E[u(c)]$$

$$p^j = \sum_s \pi_s \delta \frac{\partial u^i / \partial c_s}{\partial u^i / \partial c_0} x_s^j$$

Stochastic Discount Factor

$$p^j = \sum_s \underbrace{\pi_s \underbrace{\delta \frac{\partial u^i / \partial c_s}{\partial u^i / \partial c_0}}_{m_s}}_{q_s} x_s^j$$

- That is, stochastic discount factor $m_s = \frac{q_s}{\pi_s}$ for all s

$$p^j = \sum_s \pi_s m_s x_s^j = E[m x^j]$$

Agent's Optimization

To sum up

- Proposition 3: Suppose $c^* \gg 0$ solves problem. Then there exists positive real numbers $\lambda, m_1, \dots m_S$, such that

$$\frac{\partial U^i}{\partial c_0} = \lambda$$

$$\frac{\partial U^i}{\partial c_1} = (\mu_1, \dots, \mu_S)$$

$$\lambda p^j = \sum_s \mu_s x_s^j, \forall j = 1, \dots, J$$

(The converse is also true.)

- The vector of marginal rate of substitutions $MRS_{s,0}$ is a (positive) state price vector.

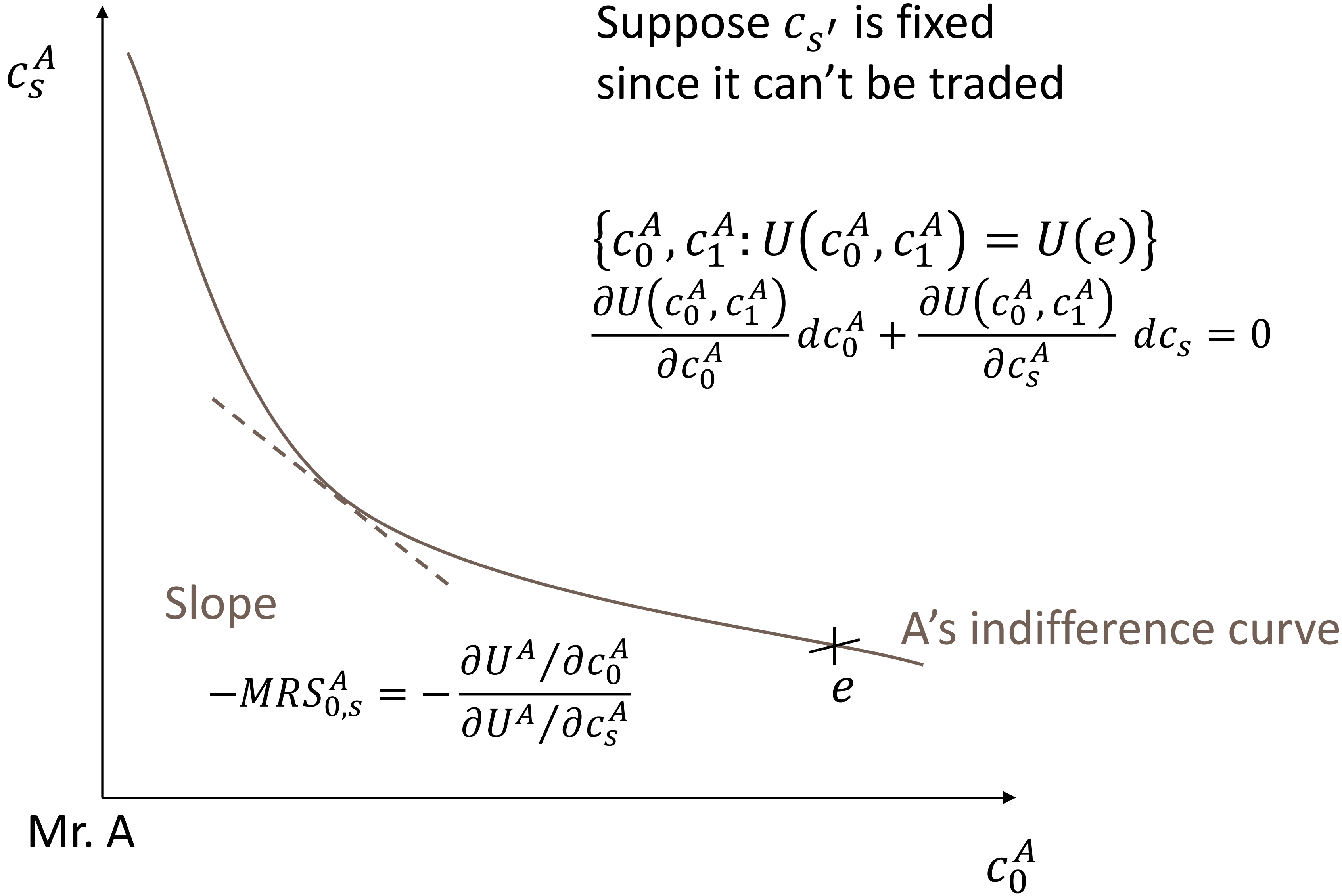
Overview

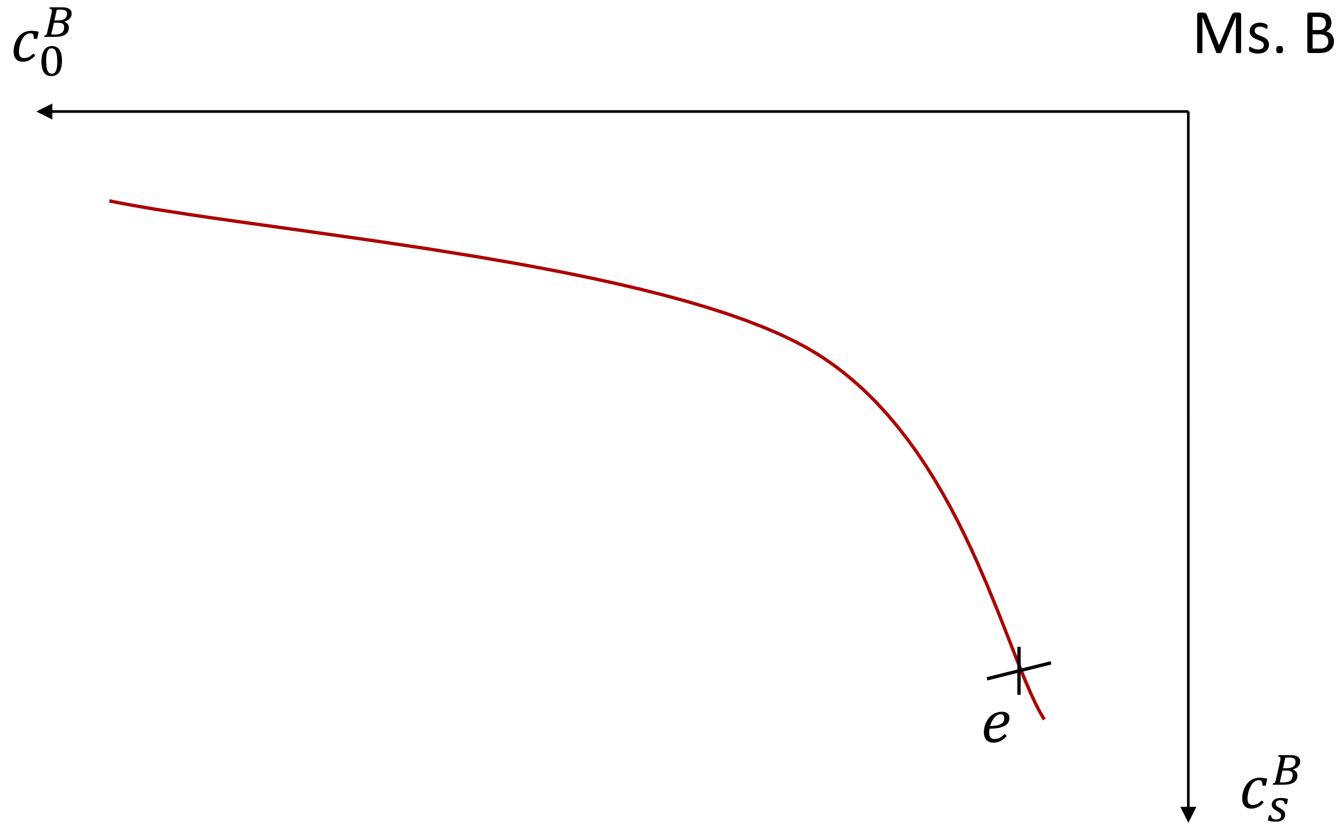
1. Marginal Rate of Substitution (MRS)
2. Pareto Efficiency
3. Welfare Theorems
4. Representative Agent Economy

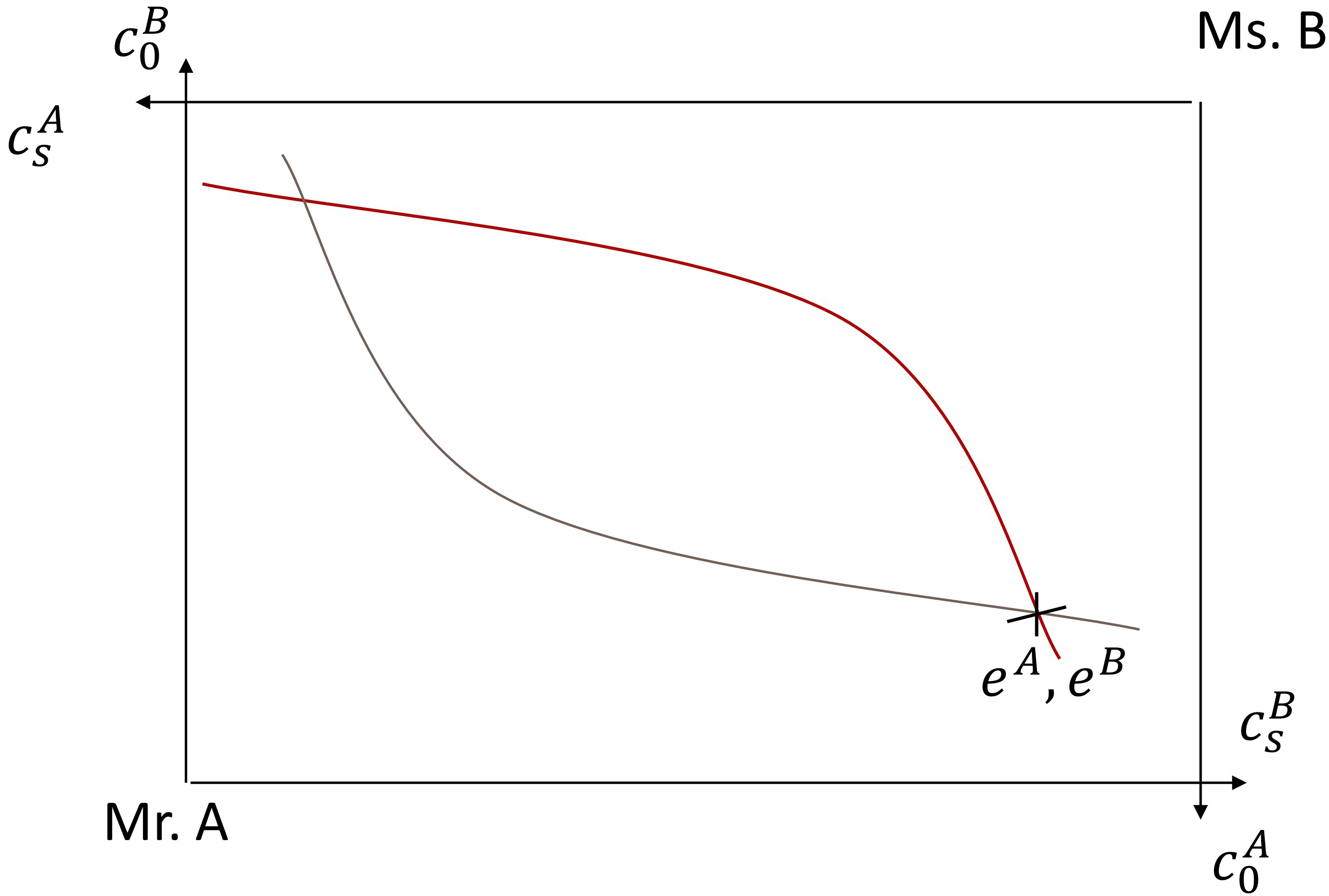
Suppose $c_{s'}$ is fixed since it can't be traded

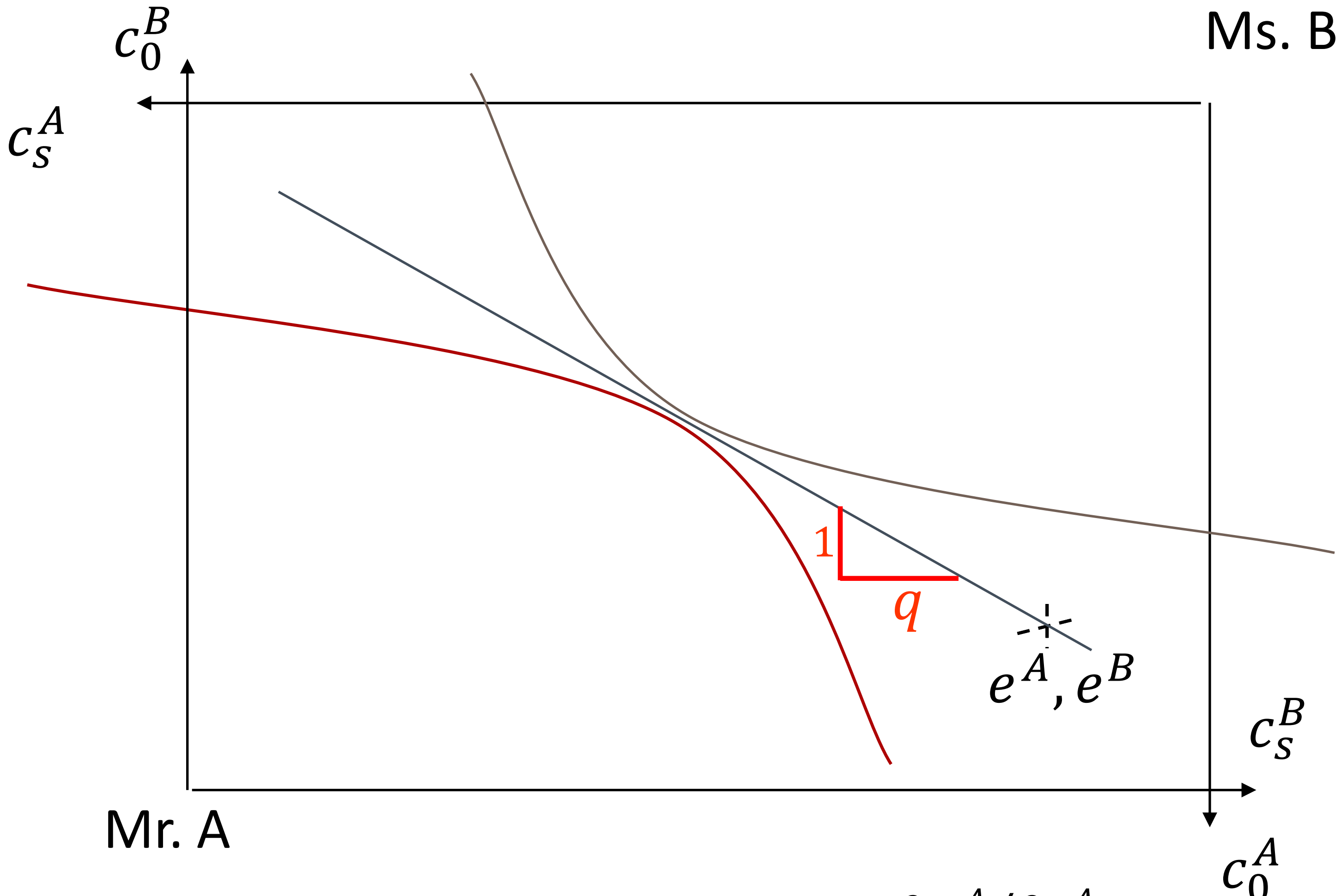
$$\left\{c_0^A, c_1^A : U\left(c_0^A, c_1^A\right) = U(e)\right\}$$

$$\frac{\partial U\left(c_0^A, c_1^A\right)}{\partial c_0^A} dc_0^A + \frac{\partial U\left(c_0^A, c_1^A\right)}{\partial c_s^A}\, dc_s = 0$$



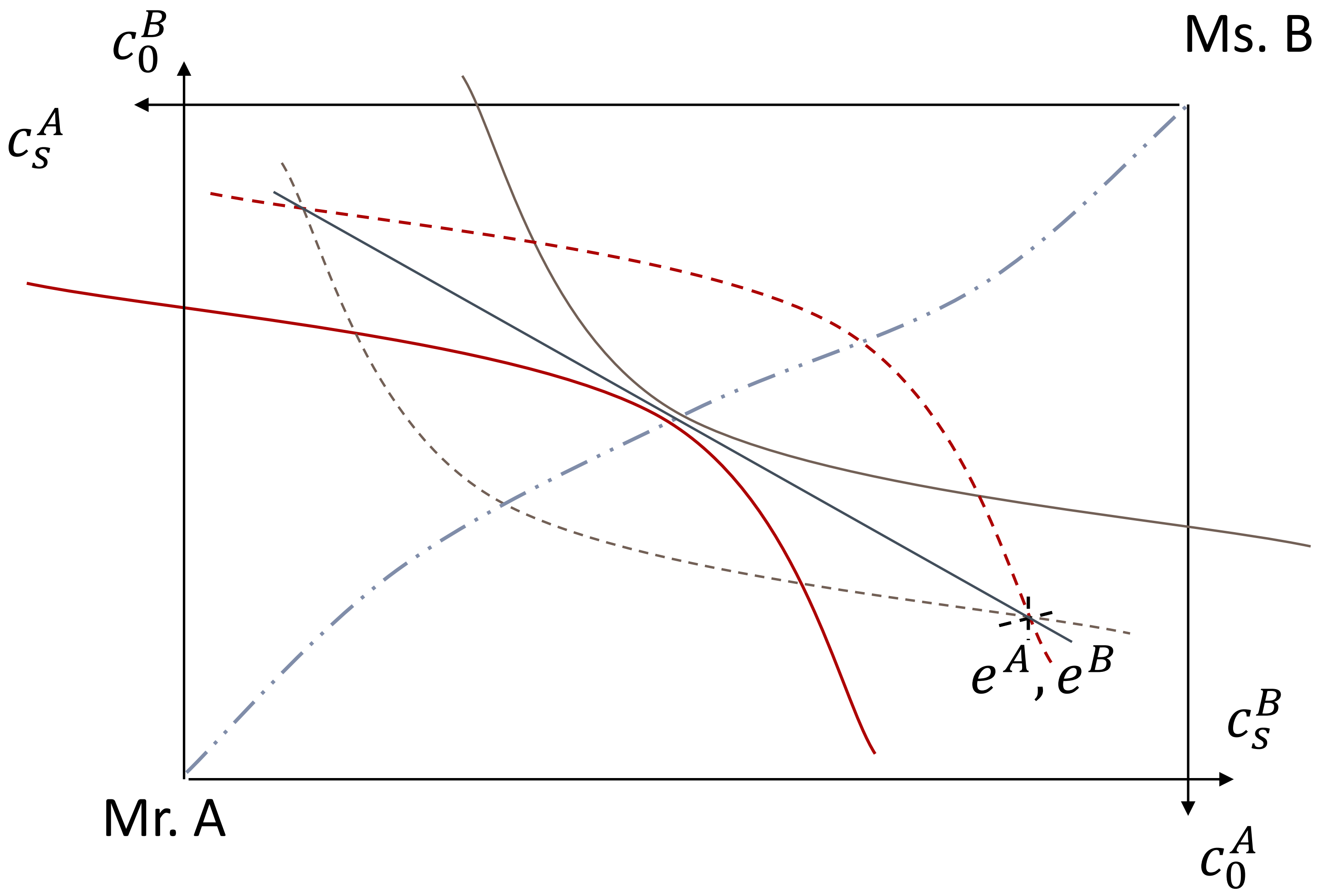

c_0^B
Ms. B
e
c_s^B


c_0^B
Ms. B
c_s^A
e^A, e^B
c_s^B
Mr. A
c_0^A



$$q = \frac{1}{MRS_{0,s}} = MRS_{s,0} = -\frac{\partial U^A/\partial c_s^A}{\partial U^A/\partial c_0^A}$$

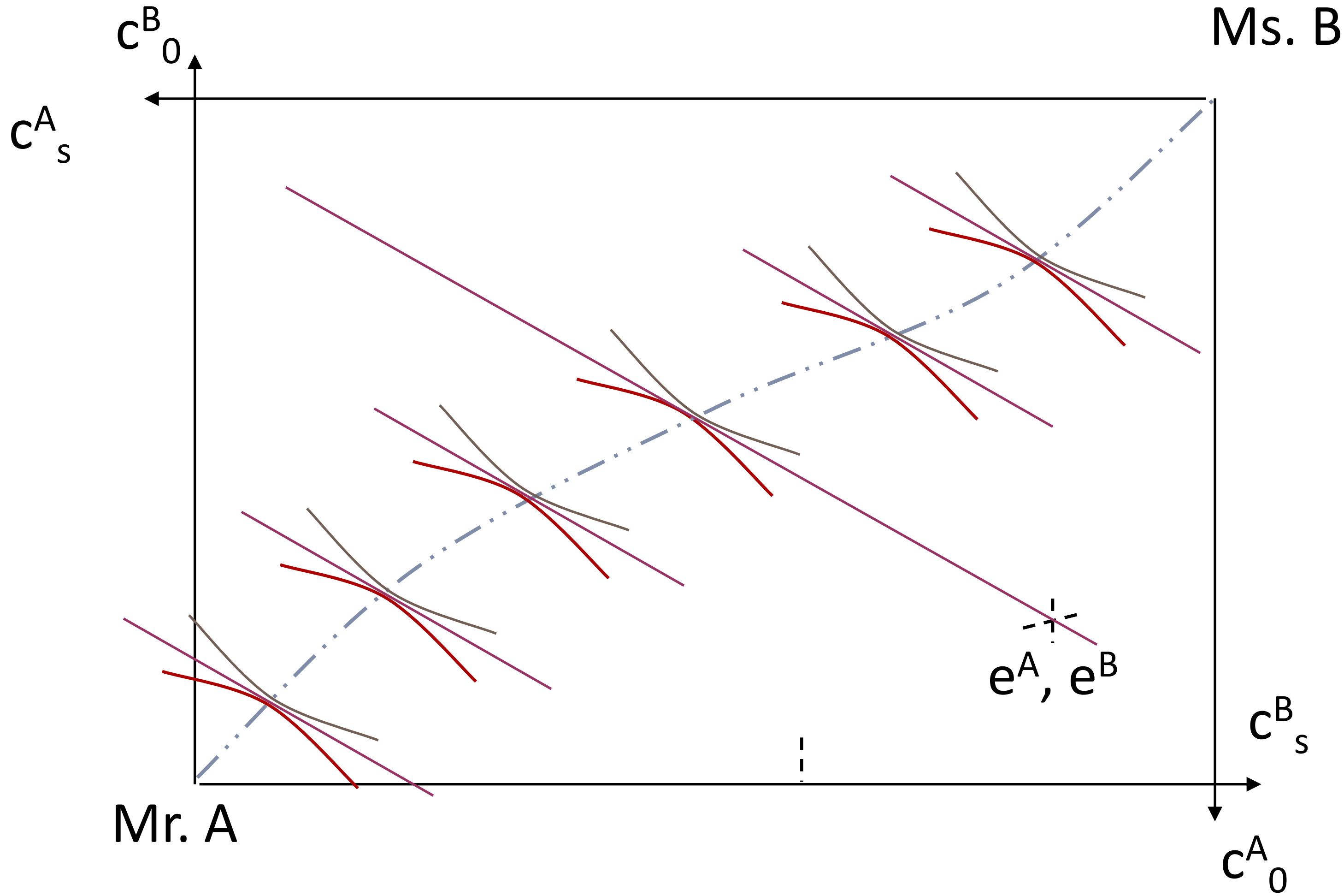
Set of PO allocations (contract curve)



Pareto Efficiency

- Allocation of resources such that
  - there is no possible redistribution such that
    - at least one person can be made better off
    - without making somebody else worse off
- Note
  - Allocative efficiency $\neq$ Informational efficiency
  - Allocative efficiency $\neq$ fairness

Set of PO allocations (contract curve)
$MRS^A = MRS^B$



Overview

1. Marginal Rate of Substitution (MRS)
2. Pareto Efficiency
3. Welfare Theorems
4. Representative Agent Economy

Welfare Theorems

- *First Welfare Theorem.* If markets are complete, then the equilibrium allocation is Pareto optimal.
  - State price is unique q. All $MRS^i(c^*)$ coincide with unique state price q.
  - Despite (pecuniary) externalities
- *Second Welfare Theorem.* Any Pareto efficient allocation can be decentralized as a competitive equilibrium.

Knife-edginess of Welfare Theorem

- In multi-period (or multiple good) setting if markets are incomplete, then equilibrium allocation is generically not only Pareto inefficient but also constrained Pareto inefficient.
  - i.e. a social planner can do better even if restricted to the same trading space
  - Pecuniary externalities can lead to wealth shifts
    - With incomplete markets not all MRS are equalized, hence pecuniary externalities generically lead to inefficiencies.
  - … more when we study multi-period settings

Overview

1. Marginal Rate of Substitution (MRS)
2. Pareto Efficiency
3. Welfare Theorems
4. Aggregation/Representative Agent Economy

Representative Agent & Complete Markets

- Aggregation Theorem 1: Suppose
  - markets are complete

  Then asset prices in economy with *many agents* are identical to an economy with a *single agent/planner* whose utility is

$$U(c) = \sum_k \alpha_k u^k(c)$$

where α^k is the welfare weight of agent k.
and the single agent consumes the aggregate endowment.

Representative Agent & HARA utility world

- Aggregation Theorem 2: Suppose
  - riskless annuity and endowments are tradable.
  - agents have common beliefs
  - agents have a common rate of time preference
  - agents have LRT (HARA) preferences with

    $R_A(c) = \frac{1}{A_i + Bc} \Rightarrow$ linear risk sharing rule

  Quasi-complete

  Then asset prices in economy with *many agents* are identical to a *single agent* economy with HARA preferences with

$$R_A(c) = \frac{1}{\sum_i A_i + Bc}$$

  - Recall fund separation theorem in Lecture 04.