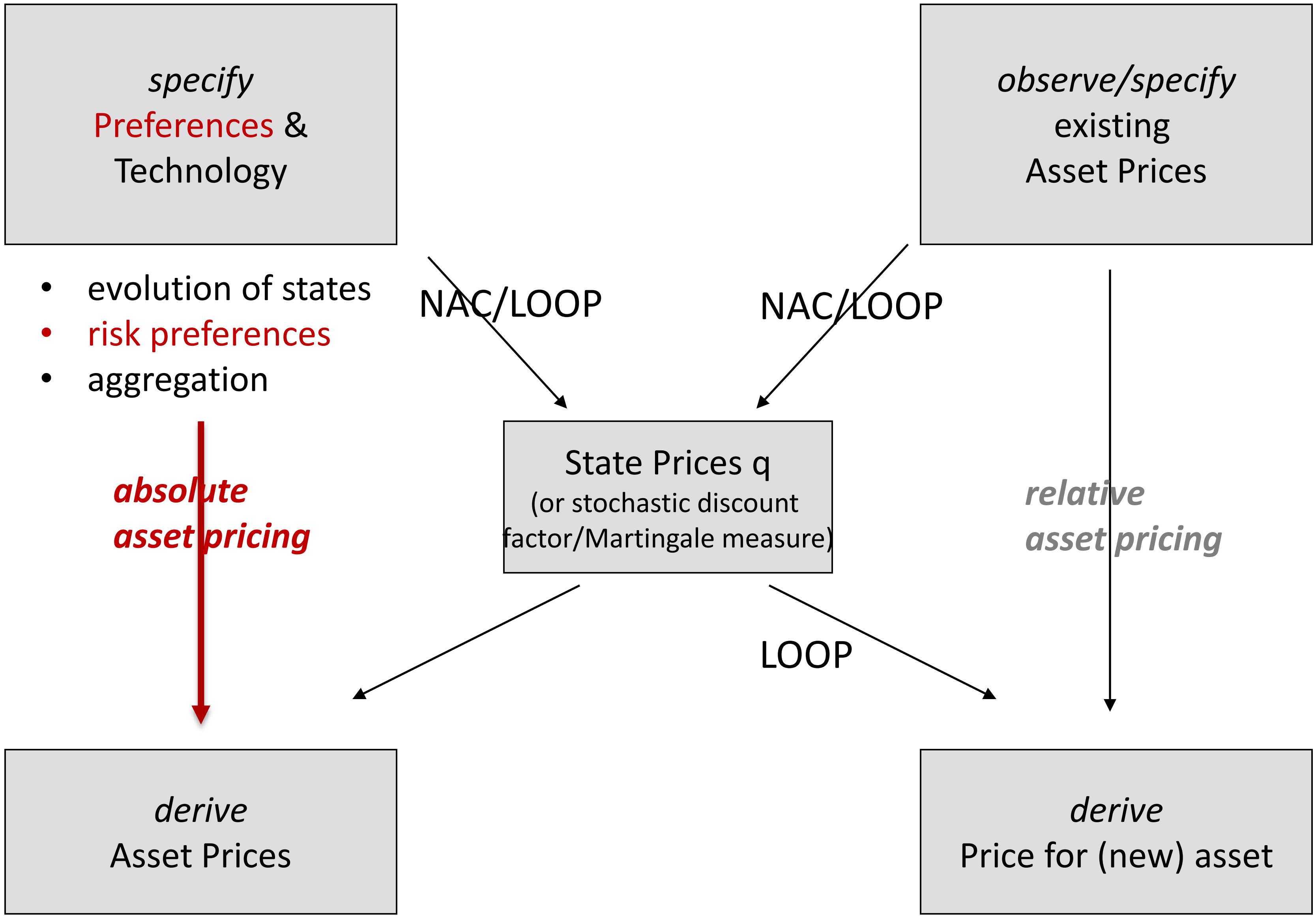
Markus K. Brunnermeier

LECTURE 4: RISK PREFERENCES & EXPECTED UTILITY THEORY



Only works as long as market completeness doesn't change

Overview: Risk Preferences

1. State-by-state dominance
2. Stochastic dominance [DD4]
3. vNM expected utility theory
   a) Intuition [L4]
   b) Axiomatic foundations [DD3]
4. Risk aversion coefficients and portfolio choice [DD5,L4]
5. Uncertainty/ambiguity aversion
6. Prudence coefficient and precautionary savings [DD5]
7. Mean-variance preferences [L4.6]

State-by-state Dominance

- State-by-state dominance is an incomplete ranking

	$t = 0$	$t = 1$	
	Cost	Payoff $\pi_1 = \pi_2 = 1/2$	
		$s = 1$	$s = 2$
Investment 1	-1000	1050	1200
Investment 2	-1000	500	1600
Investment 3	-1000	1050	1600

- Investment 3 state-by-state dominates investment 1
- Recall: $y, x \in \mathbb{R}^S$
  - $y \geq x \Leftrightarrow y_s \geq x_s$ for each $s = 1, \ldots, S$
  - $y > x \Leftrightarrow y \geq x, y \neq x$
  - $y \gg x \Leftrightarrow y_s > x_s$ for each $s = 1, \ldots, S$

State-by-state Dominance (ctd.)

	$t = 0$	$t = 1$			
	Cost	Return $\pi_1 = \pi_2 = 1/2$		E[Return]	σ
		$s = 1$	$s = 2$		
Investment 1	-1000	+ 5%	+ 20%	12.5%	7.5%
Investment 2	-1000	- 50%	+ 60%	5%	55%
Investment 3	-1000	+ 5%	+ 60%	32.5%	27.5%

- Investment 1 mean-variance dominates 2
- But, investment 3 does *not* mean-variance dominate 1

State-by-state Dominance (ctd.)

	$t=0$	$t=1$			
	Cost	Return $\pi_1 = \pi_2 = 1/2$		E[Return]	σ
		$s=1$	$s=2$		
Investment 4	-1000	+ 3%	+ 5%	4.0%	1.0%
Investment 5	-1000	+ 3%	+ 8%	5.5%	2.5%

- What is the trade-off between risk and expected return?
- Investment 4 has a higher Sharpe ratio $\frac{E[r]-r_f}{\sigma}$ than investment 5 for $r_f = 0$

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Stochastic Dominance

- No state-space – probabilities are not assigned specific states
  - Only applicable for final payoff gamble
    - Not for stocks/lotteries that form a portfolio (whose payoff is final)
  - Random variables before introduction of $(\Omega, \mathcal{F}, P)$
- Still incomplete ordering
  - "More complete" than state-by-state ordering
  - State-by-state dominance ⇒ stochastic dominance
  - Risk preference not needed for ranking!
    - independently of the specific trade-offs (between return, risk and other characteristics of probability distributions) represented by an agent's utility function. ("risk-preference-free")
- Next Section:
  - Complete preference ordering and utility representations

From payoffs per state to probability

state	s_1	s_2	s_3	s_4	s_5
probability	π_1	π_2	π_3	π_4	π_5
Payoff x	10	10	20	20	20
Payoff y	10	20	20	20	30

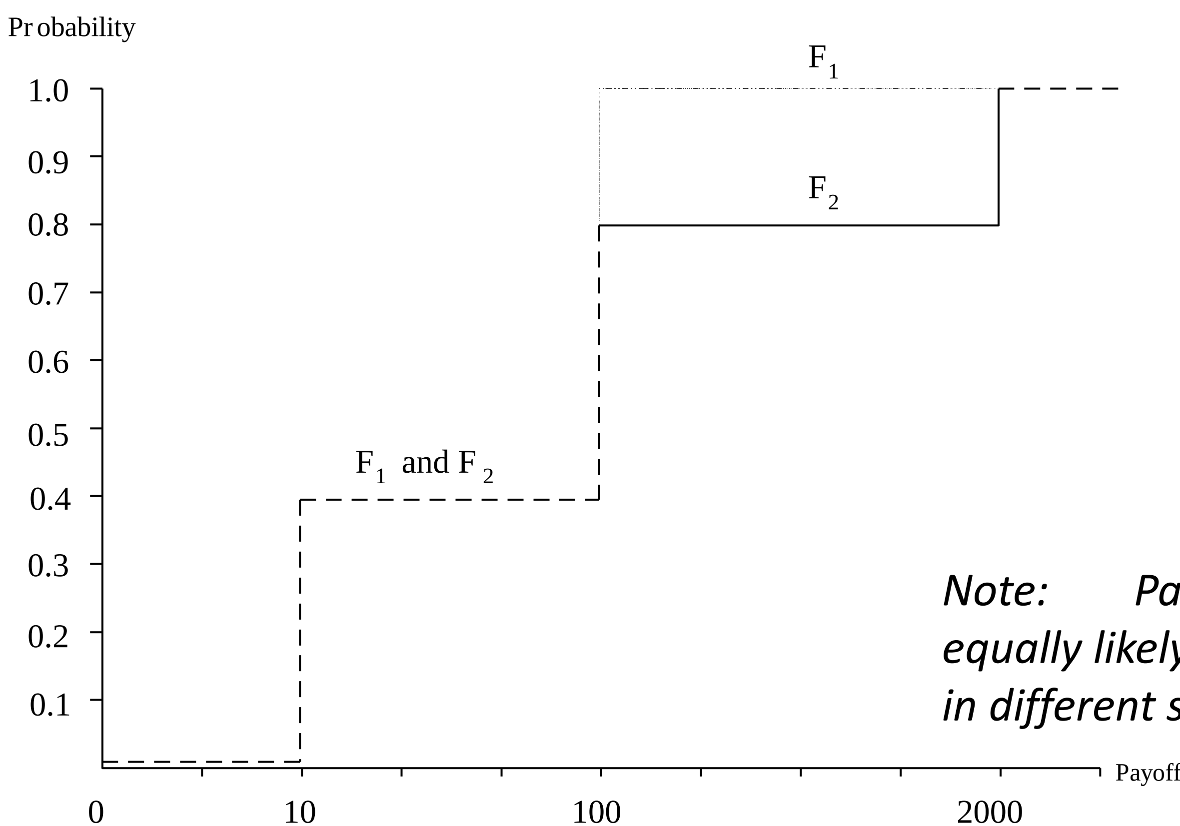
Expressed in "probability lotteries" – only useful for final payoffs

(since some cross correlation information ins lost)

payoff	**10**	**20**	**30**
Prob x	$p_{10} = \pi_1 + \pi_2$	$p_{20} = \pi_3 + \pi_4 + \pi_5$	$p_{30} = 0$
Prob y	$q_{10} = \pi_1$	$q_{20} = \pi_2 + \pi_3 + \pi_4$	$q_{30} = \pi_5$

Preference $x \succ y \in \mathbb{R}^S$ expressed in probabilities $p_x \succ q_y$

				Expected Payoff	Standard Deviation
Payoffs	10	100	2000		
Probability 1	.4	.6	0	64	44
Probability 2	.4	.4	.2	444	779



Note: Payoff of $10 is equally likely, but can be in different states of the world

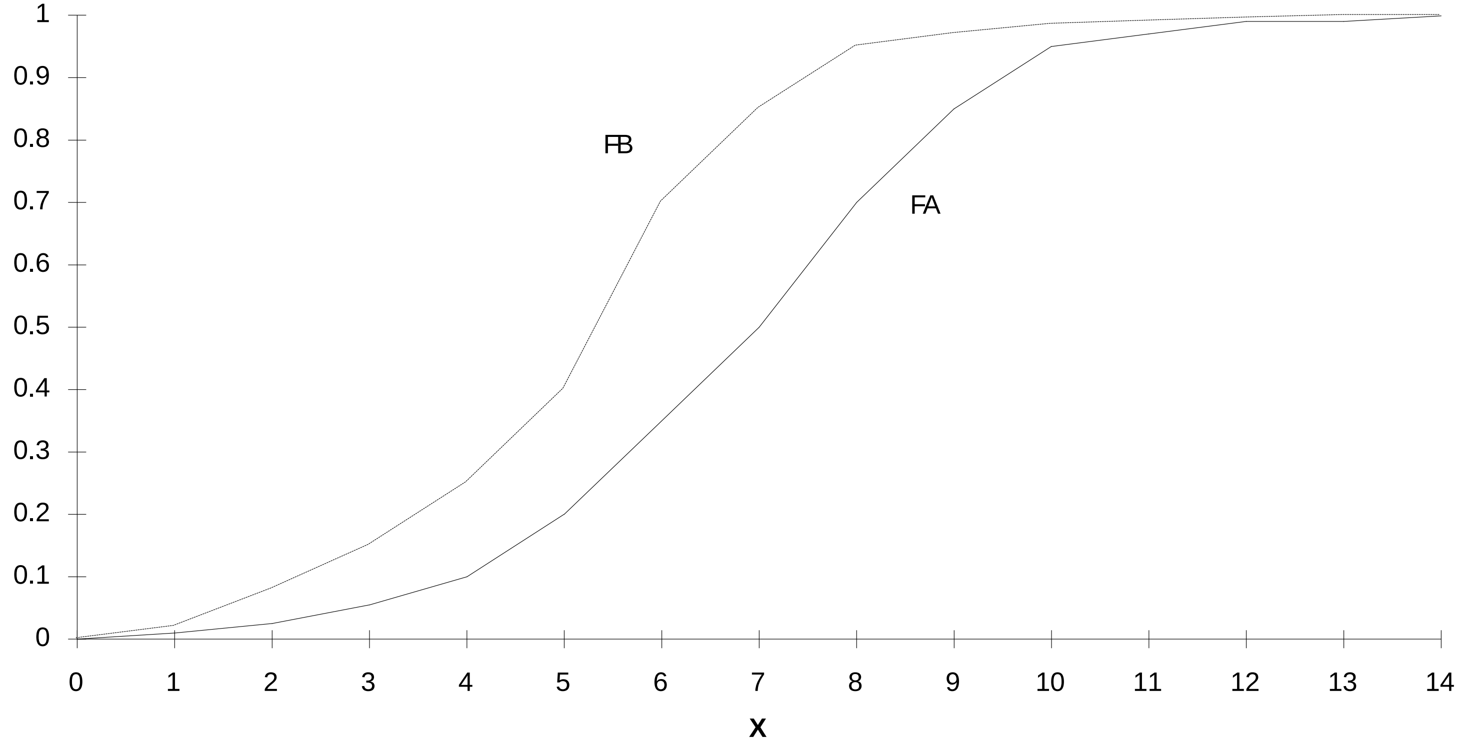
First Order Stochastic Dominance

- Definition: *Let* $F_A(x), F_B(x)$*, respectively, represent the cumulative distribution functions of two random variables (cash payoffs) that, without loss of generality assume values in the interval* $[a, b]$*. We say that* $F_A(x)$ *first order stochastically dominates (FSD)* $F_B(x)$ *if and only if for all* $x \in [a, b]$

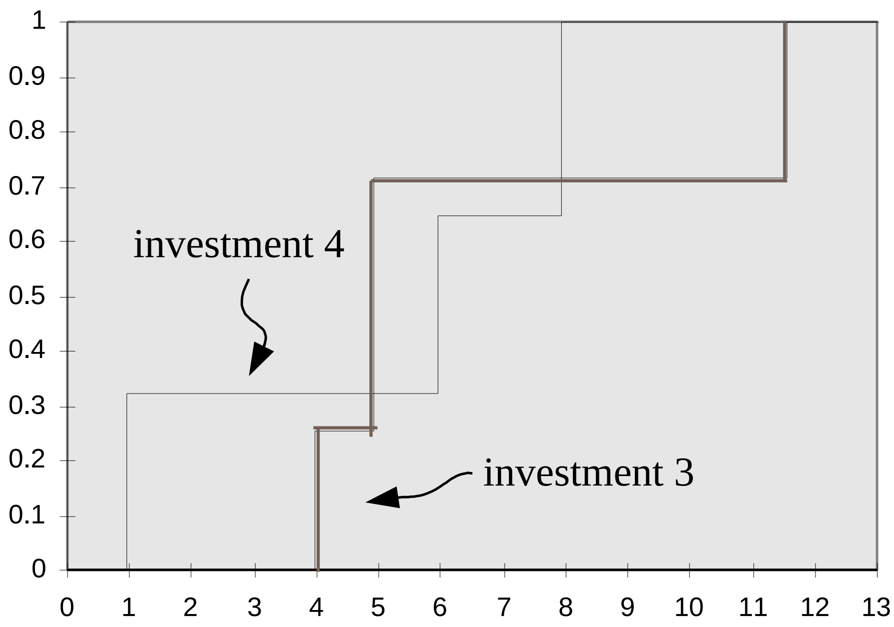
$$F_A(x) \leq F_B(x)$$

Homework: Provide an example which can be ranked according to FSD, but not according to state dominance.

First Order Stochastic Dominance



Payoff	1	4	5	6	8	12
Probability 3	0	.25	0.50	0	0	.25
Probability 4	.33	0	0	.33	.33	0



CDFs of investment 3 and 4

Second Order Stochastic Dominance

- Definition: *Let* $F_A(x), F_B(x)$ *be two cumulative probability distribution for random payoffs in* $[a, b]$. *We say that* $F_A(x)$ *second order stochastically dominates* (SSD) $F_B(x)$ *if and only if for any* $x \in [a, b]$

$$\int_{-\infty}^{x} [F_B(t) - F_A(t)]dt \geq 0$$

(with strict inequality for some meaningful interval of values of *t*).

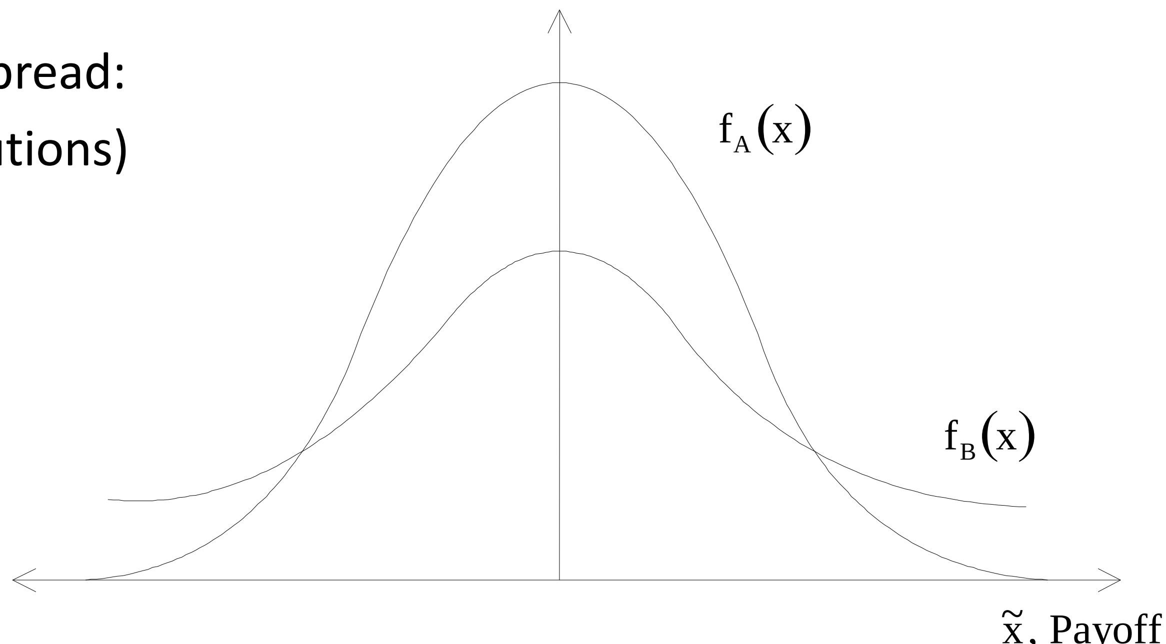
Mean Preserving Spread

$$x_B = x_A + z$$

(where z is independent and has zero mean)

Mean Preserving Spread:

(for normal distributions)



$$\mu = \int x\, f_A(x) dx = \int x\, f_B(x) dx$$

Mean Preserving Spread & SSD

- Theorem: *Let $F_A(x)$ and $F_B(x)$ be two distribution functions defined on the same state space with identical means.*
  Then the following statements are equivalent :
  - $F_A(x)$ SSD $F_B(x)$
  - $F_B(x)$ is a mean-preserving spread of $F_A(x)$

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A Hypothetical Gamble

- Suppose someone offers you this gamble:
  - "I have a fair coin here. I'll flip it, and if it's tails I pay you $1 and the gamble is over. If it's heads, I'll flip again. If it's tails then, I pay you $2, if not I'll flip again. With every round, I double the amount I will pay to you if it turns up tails."
- Sounds like a good deal. After all, you can't lose. So here's the question:
  - **How much are you willing to pay to take this gamble?**

Proposal 1: Expected Value

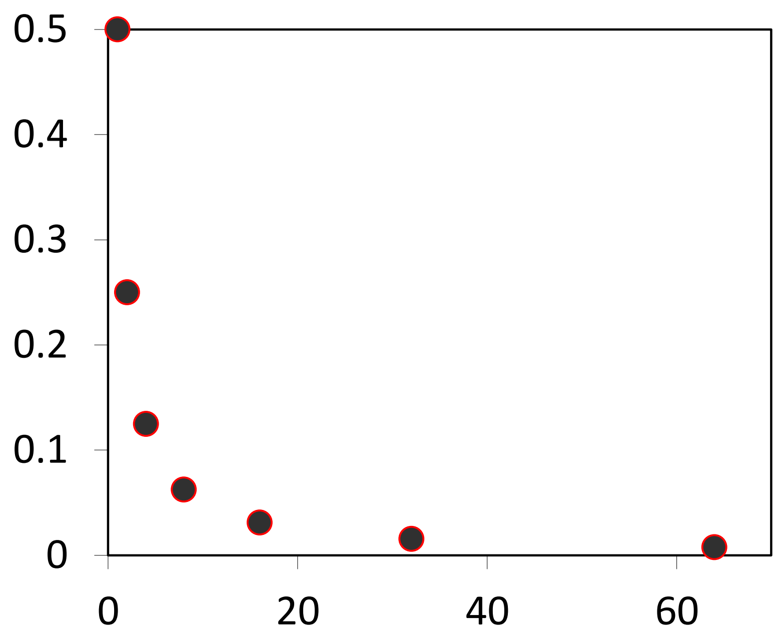
- With probability $\frac{1}{2}$ you get \$1, $\left(\frac{1}{2}\right)^1 \times 2^0$
- With probability $\frac{1}{4}$ you get \$2, $\left(\frac{1}{2}\right)^2 \times 2^1$
- With probability $\frac{1}{8}$ you get \$4, $\left(\frac{1}{2}\right)^3 \times 2^2$
- …

  - The expected payoff is given by the sum of all these terms, i.e.

$$\sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t \times 2^{t-1} = \sum_{t=1}^{\infty} \frac{1}{2} = \infty$$

St. Petersburg Paradox

- You should pay everything you own *and more* to purchase the right to take this gamble!
- Yet, in practice, no one is prepared to pay such a high price. Why?
- Even though the expected payoff is infinite, the distribution of payoffs is not attractive…
  - with 93% probability we get $8 or less;
  - with 99% probability we get $64 or less



Proposal 2

- Bernoulli suggests that large gains should be weighted less. He suggests to use the natural logarithm.
  [Cremer - another great mathematician of the time - suggests the square root.]

$$\sum_{t=1}^{\infty}\left(\frac{1}{2}\right)^{t} \times \ln 2^{t-1} = \ln 2 < \infty$$

According to this Bernoulli would pay at most $e^{\ln 2} = 2$ to participate in this gamble

Representation of Preferences

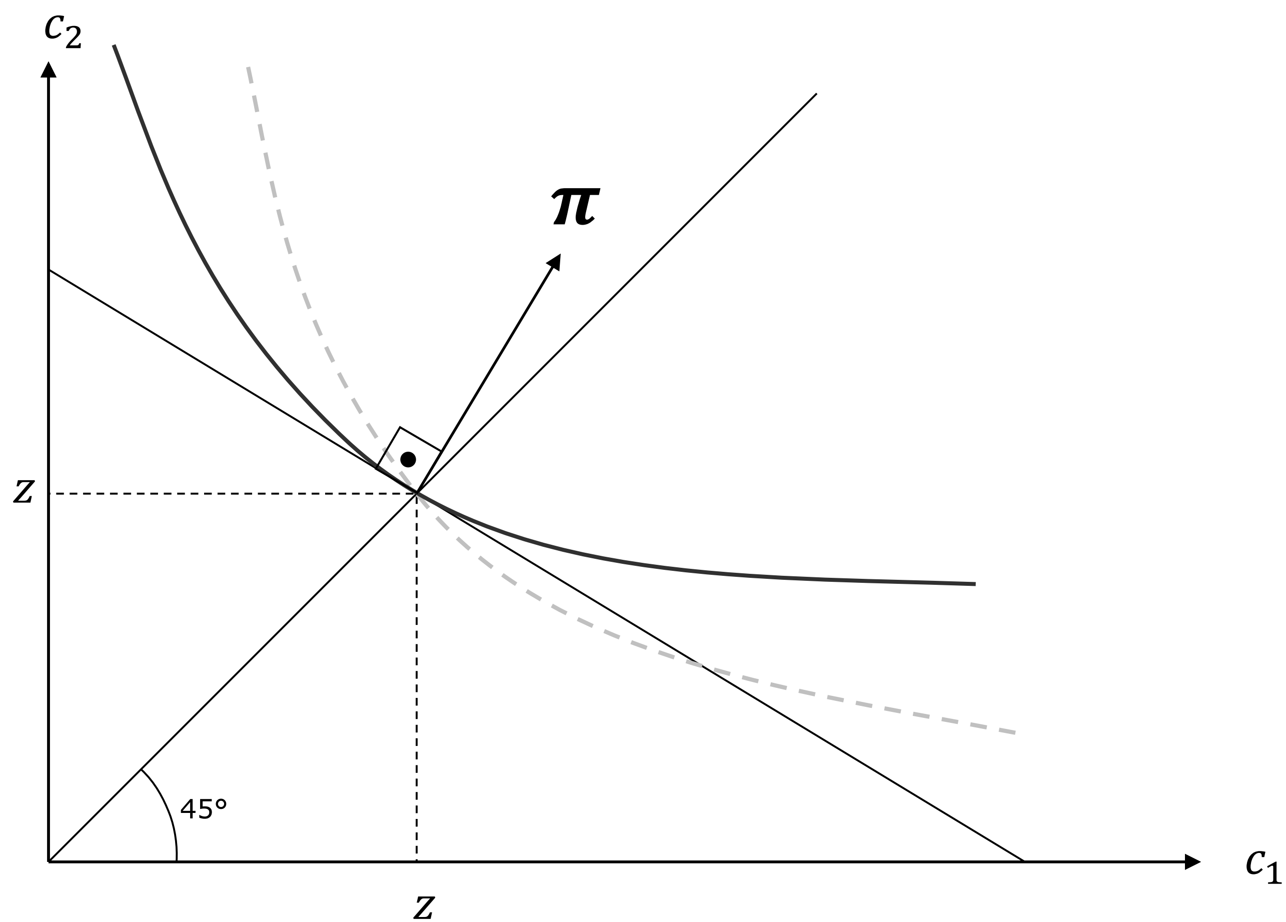
A preference ordering is (i) complete, (ii) transitive, (iii) continuous and [(iv) relatively stable] can be represented by a utility function, i.e.

$$(c_0, c_1, \dots, c_S) \succ (c'_0, c'_1, \dots, c'_S)$$
$$\Leftrightarrow U(c_0, c_1, \dots, c_S) > U(c'_0, c'_1, \dots, c'_S)$$

(preference ordering *over lotteries* –
$(S+1)$-dimensional space)

Indifference curves

in $\mathbb{R}^2$ (for $S = 2$)



Preferences over Prob. Distributions

- Consider c_0 fixed, c_1 is a random variable
- Preference ordering over probability distributions
- Let
  - P be a set of probability distributions with a finite support over a set X,
  - $\succcurlyeq$ preference ordering over P (that is, a subset of $P \times P$)

Prob. Distributions

- *S* states of the world
- Set of all possible lotteries

$$P = \{p \in \mathbb{R}^S | p(c) \geq 0, \sum p(c) = 1\}$$

- Space with *S* dimensions

- Can we simplify the utility representation of preferences over lotteries?
- Space with *one* dimension – income
- We need to assume further axioms

Expected Utility Theory

- A binary relation that satisfies the following three axioms if and only if there exists a function $u(\cdot)$ such that

$$p \succ q \Leftrightarrow \sum p(c)u(c) > \sum q(c)u(c)$$

i.e. preferences correspond to expected utility.

vNM Expected Utility Theory

- Axiom 1 (Completeness and Transitivity):
  - Agents have preference relation over P (repeated)
- Axiom 2 (Substitution/Independence)
  - For all lotteries $p, q, r \in P$ and $\alpha \in (0,1]$,
    $p \succcurlyeq q \Leftrightarrow \alpha p + (1-\alpha) r \succcurlyeq \alpha q + (1-\alpha) r$
- Axiom 3 (Archimedian/Continuity)
  - For all lotteries $p, q, r \in P$ if $p \succ q \succ r$ then there exists $\alpha, \beta \in (0,1)$ such that,

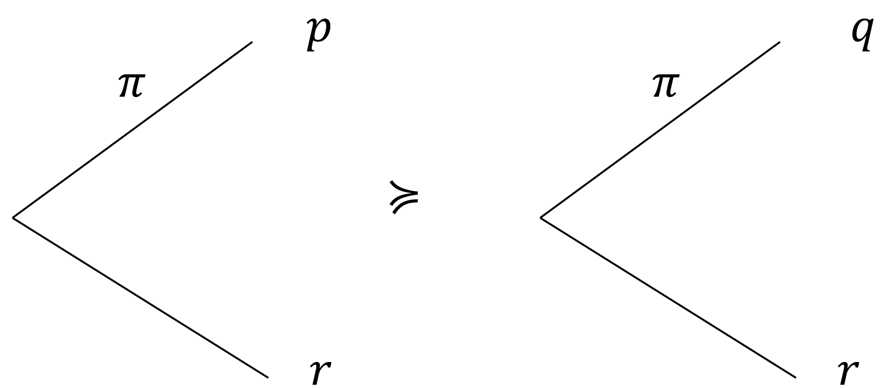
$$\alpha p + (1-\alpha) r \succ q \succ \beta p + (1-\beta) r$$

Problem: p you get \$100 for sure, q you get \$ 10 for sure, r you are killed

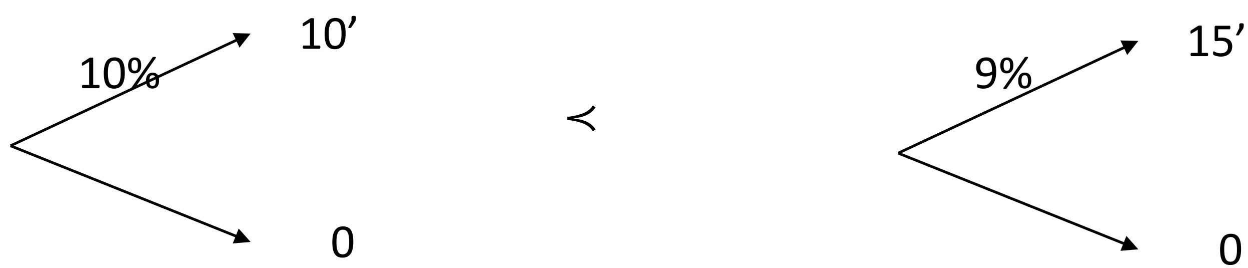
Independence Axiom

- Independence of irrelevant alternatives:

$p \succcurlyeq q \Leftrightarrow$



Allais Paradox – Violation of Independence Axiom



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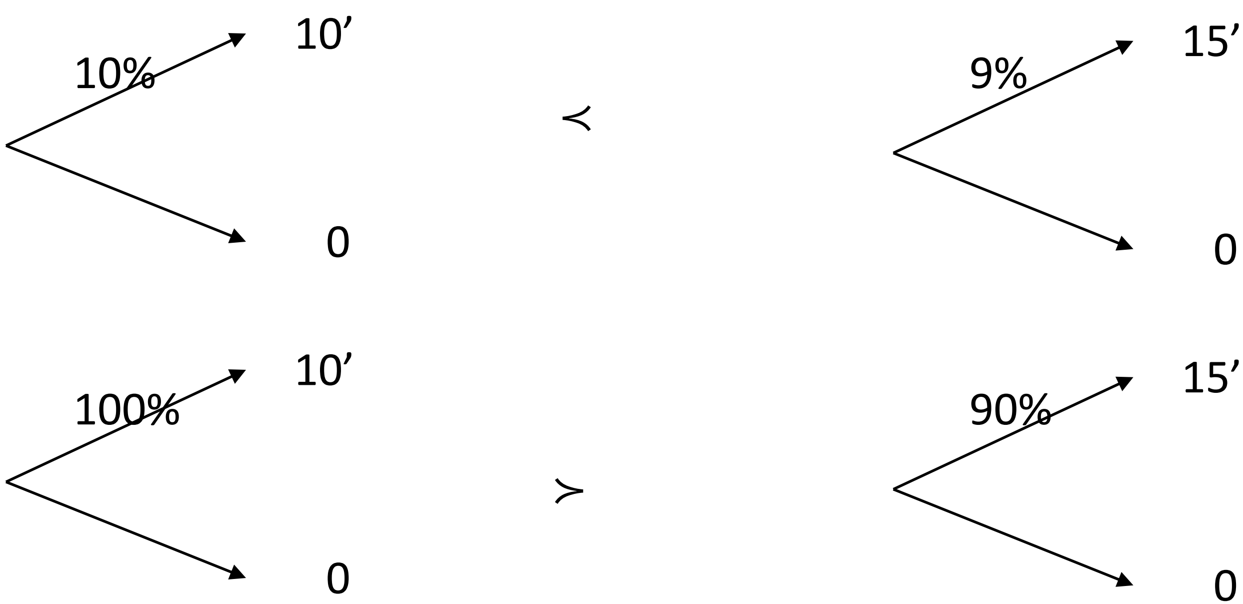

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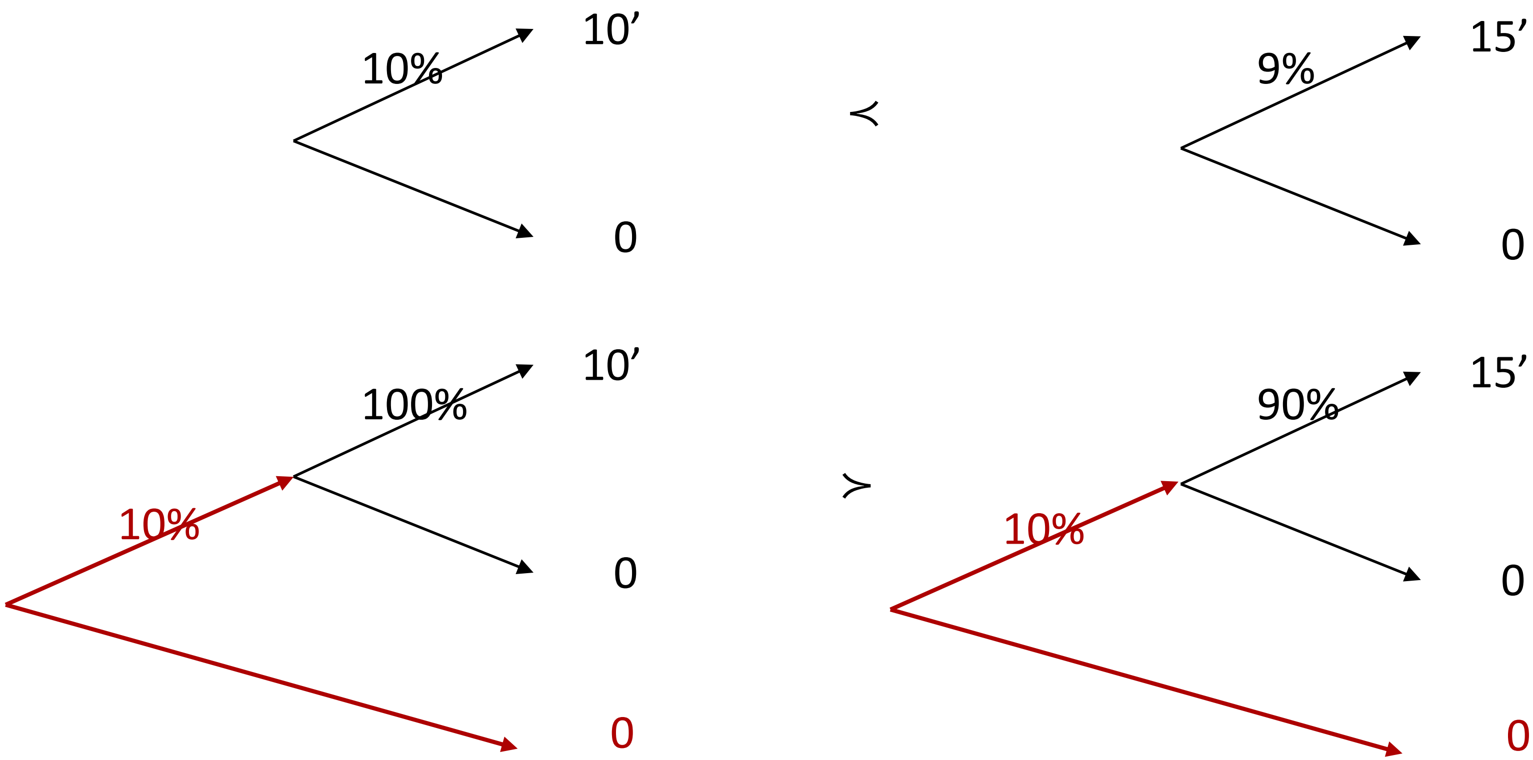


vNM EU Theorem

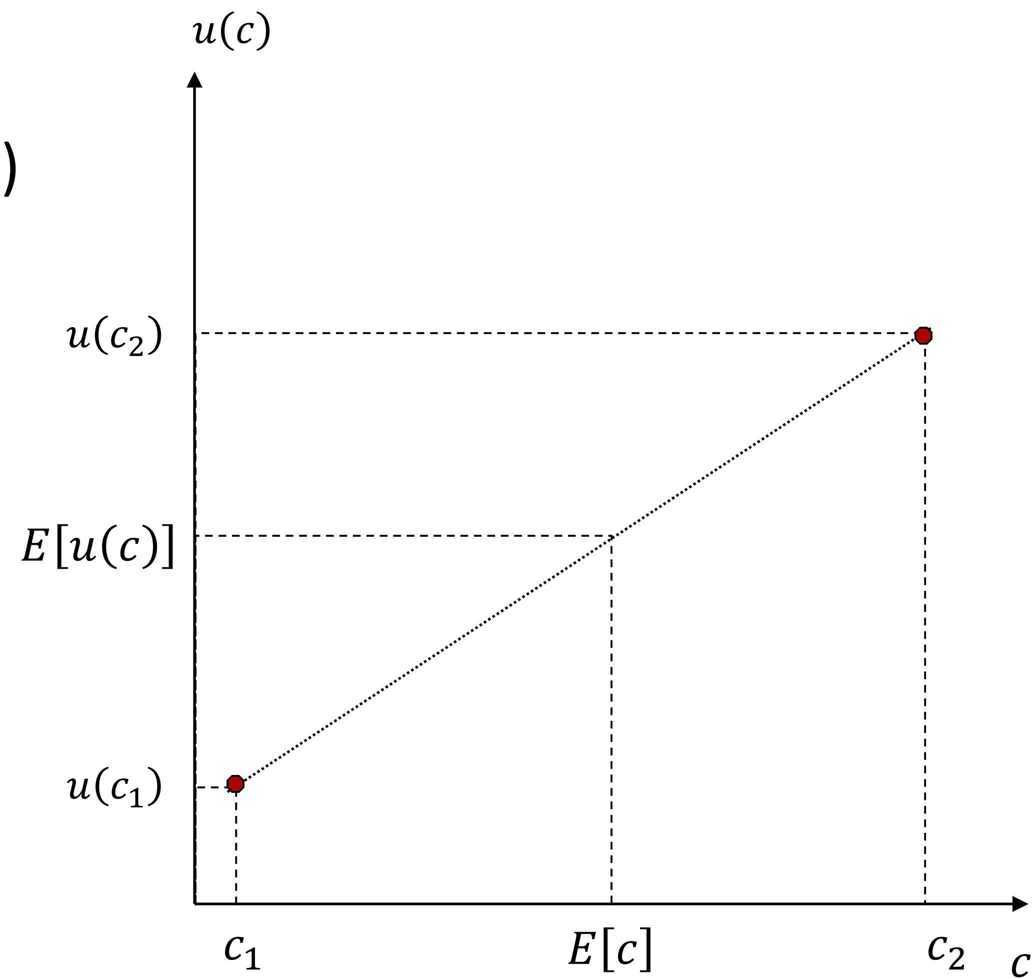
- A binary relation that satisfies the axioms 1-3 if and only if there exists a function $u(\cdot)$ such that

$$p \succ q \Leftrightarrow \sum p(c)u(c) > \sum q(c)u(c)$$

i.e. preferences correspond to expected utility.

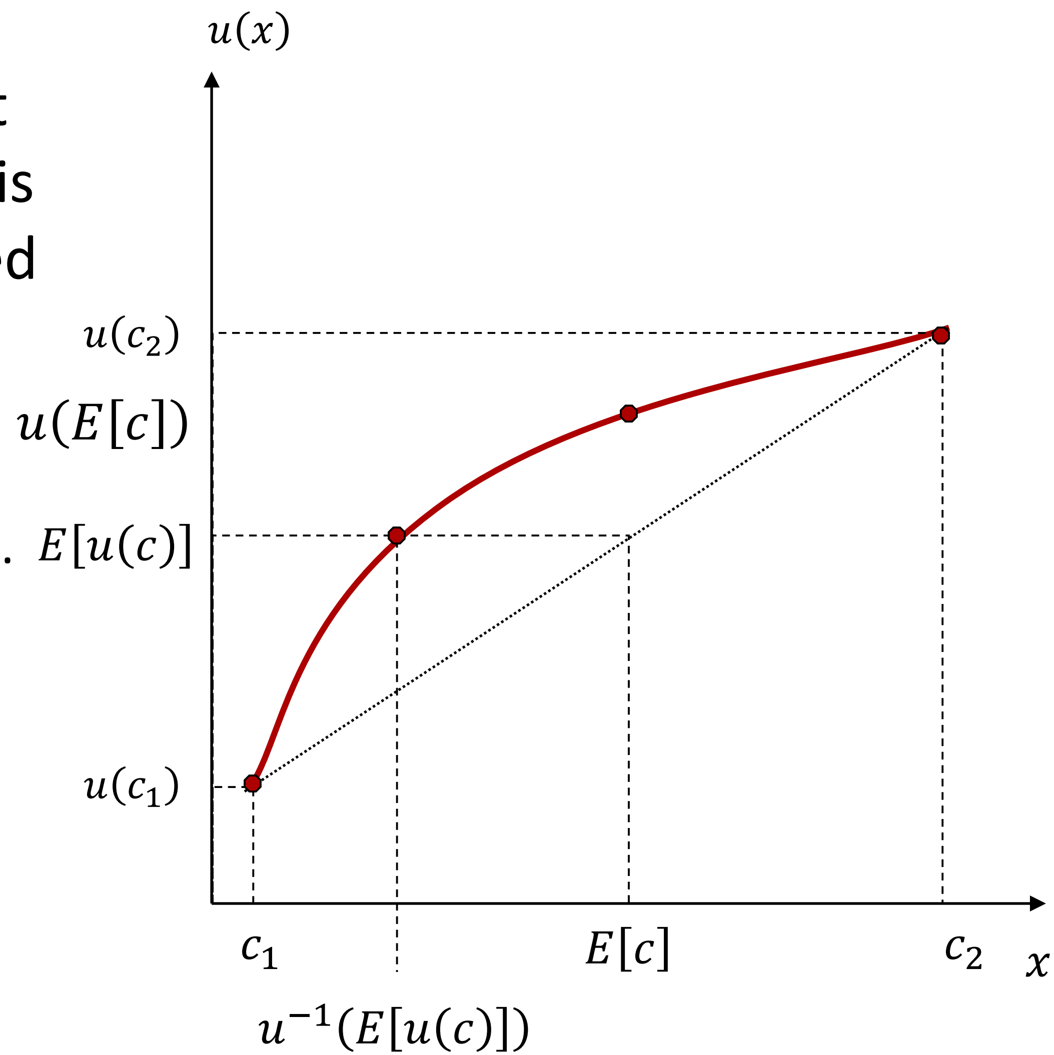
Risk-Aversion and Concavity

- The shape of the von Neumann Morgenstern (NM) utility function reflects risk preference
- Consider lottery with final wealth of c_1 or c_2



Risk-aversion and concavity

- Risk-aversion means that the certainty equivalent is smaller than the expected prize.
  - We conclude that a risk-averse vNM utility function must be concave.



Jensen's Inequality

Theorem:

- *Let* $g(\cdot)$ *be a concave function on the interval* $[a, b]$*, and* x *be a random variable such that*

$$P\big[x \in [a, b]\big] = 1$$

- *Suppose the expectations* $E[x]$ *and* $E[g(x)]$ *exist; then*

$$E[g(x)] \leq g\big[E[x]\big]$$

Furthermore, if $g(\cdot)$ *is strictly concave, then the inequality is strict.*

Expected Utility & Stochastic Dominance

- Theorem: *Let* $F_A(\tilde{x}), F_B(\tilde{x})$ *be two cumulative probability distribution for random payoffs* $\tilde{x} \in [a, b]$. *Then* $F_A(\tilde{x})$ *FSD* $F_B(\tilde{x})$ *if and only if* $E_A[u(\tilde{x})] \geq E_B[u(\tilde{x})]$ *for all non decreasing utility functions* $U(\cdot)$.

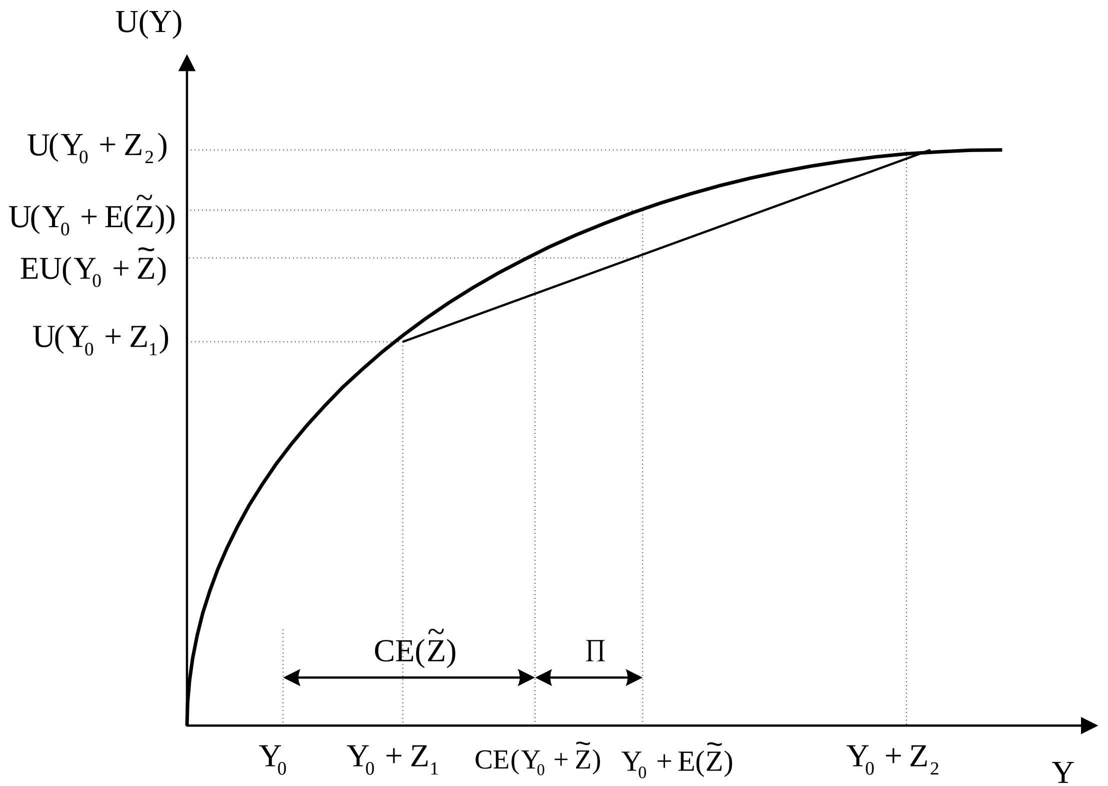
- Theorem: *Let* $F_A(\tilde{x}), F_B(\tilde{x})$ *be two cumulative probability distribution for random payoffs* $\tilde{x} \in [a, b]$. *Then* $F_A(\tilde{x})$ *SSD* $F_B(\tilde{x})$ *if and only if* $E_A[u(\tilde{x})] \geq E_B[u(\tilde{x})]$ *for all non decreasing concave utility functions* $U(\cdot)$.

Certainty Equivalent and Risk Premium

$$E\left[u\left(c+\tilde{Z}\right)\right] = u\left(c + CE\left(c,\tilde{Z}\right)\right)$$

$$E\left[u\left(c+\tilde{Z}\right)\right] = u\left(c + E[\tilde{Z}] - \Pi\left(c,\tilde{Z}\right)\right)$$

Certainty Equivalent and Risk Premium



Certainty Equivalent and Risk Premium

Utility Transformations

- General utility function:
  - Suppose $U(c_0, c_1, \dots, c_S) > U(c_0', c_1', \dots, c_S')$ represents complete, transitive,… preference ordering,
  - then $V(\cdot) = f\big(U(\cdot)\big)$, where $f(\cdot)$ is strictly increasing represents the same preference ordering
- vNM utility function
  - Suppose $E[u(c)]$ represents preference ordering satisfying vNM axioms,
  - then $v(c) = a + bu(c)$ represents the same. "affine transformation"

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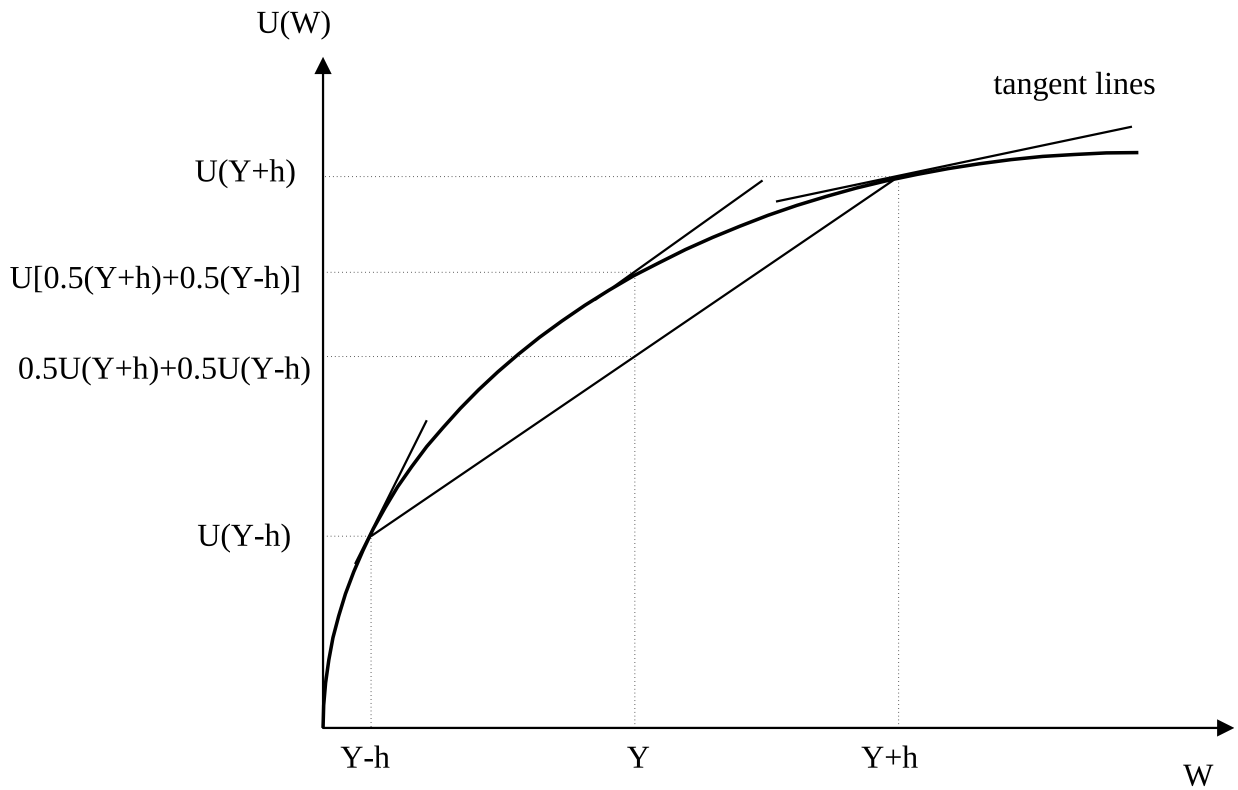
Measuring Risk aversion



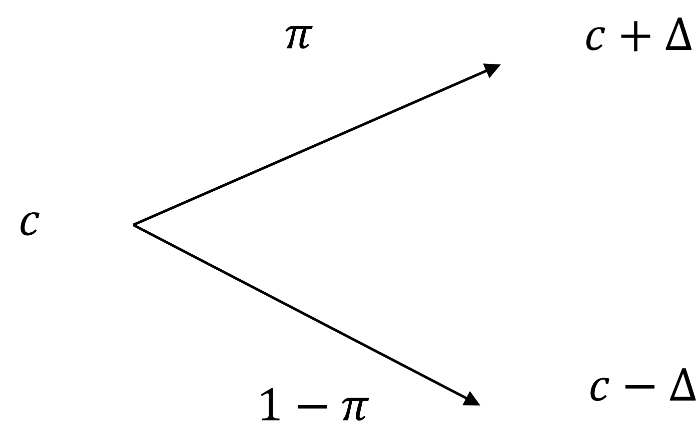
A Strictly Concave Utility Function

Arrow-Pratt Measures of Risk aversion

- absolute risk aversion $= -\frac{u''(c)}{u'(c)} \equiv R_A(c)$
- relative risk aversion $= -\frac{cu''(c)}{u'(c)} \equiv R_R(c)$
- risk tolerance $= \frac{1}{R_A}$

Absolute risk aversion coefficient

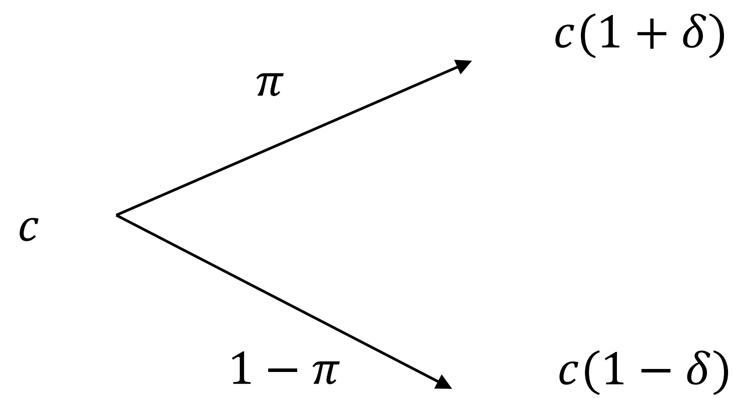
$$R_A = -\frac{u''(c)}{u'(c)}$$



$$\pi(c, \Delta) = \frac{1}{2} + \frac{1}{4}\Delta R_A(c) + HOT$$

Relative risk aversion coefficient

$$R_R = -\frac{u''(c)}{u'(c)}c$$



$$\pi(c,\theta) = \frac{1}{2} + \frac{1}{4}\delta R_R(c) + HOT$$

Homework: Derive this result.

CARA and CRRA-utility functions

- Constant Absolute RA utility function

$$u(c) = -e^{-\rho c}$$

- Constant Relative RA utility function

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \gamma \neq 1$$

$$u(c) = \ln[c], \gamma = 1$$

Level of Relative Risk Aversion

$$\frac{(Y+CE)^{1-\gamma}}{1-\gamma}=\frac{\frac{1}{2}(Y+50000)^{1-\gamma}}{1-\gamma}+\frac{\frac{1}{2}(Y+100000)^{1-\gamma}}{1-\gamma}$$

$\underline{Y=0}$	$\gamma = 0$	CE = 75,000 (risk neutrality)
	$\gamma = 1$	CE = 70,711
	$\gamma = 2$	CE = 66,246
	$\gamma = 5$	CE = 58,566
	$\gamma = 10$	CE = 53,991
	$\gamma = 20$	CE = 51,858
	$\gamma = 30$	CE = 51,209
$\underline{Y=100000}$	$\gamma = 5$	CE = 66,530

Risk aversion and Portfolio Allocation

- No savings decision (consumption occurs only at *t=1*)
- Asset structure
  - One risk free bond with net return r_f
  - One risky asset with random net return r
    (a =quantity of risky assets)

$$\max_a E\left[u\left(Y_0(1+r_f)+a(r-r_f)\right)\right]$$

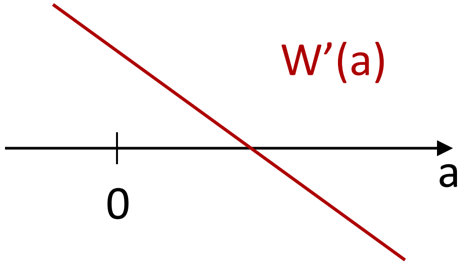
$$\text{FOC} \Rightarrow E\left[u'\left(Y_0(1+r_f)+a(r-r_f)\right)(r-r_f)\right] = 0$$

Risk aversion and Portfolio Allocation

- Theorem 4.1: Assume $U' > 0, U'' < 0$ and let $\hat{a}$ denote the solution to above problem. Then

$$\hat{a} > 0 \Leftrightarrow E[\hat{r}] > r_f$$
$$\hat{a} = 0 \Leftrightarrow E[\tilde{r}] = r_f$$
$$\hat{a} < 0 \Leftrightarrow E[\tilde{r}] < r_f$$

- Define $W(a) = E\left[u\left(Y_0(1 + r_f) + a(\tilde{r} - r_f)\right)\right]$. The FOC can then be written $W'(a) = E\left[u'\left(Y_0(1 + r_f) + a(\tilde{r} - r_f)\right)(\tilde{r} - r_f)\right] = 0$.
- By risk aversion $W''(a) = E\left[u''\left(Y_0(1 + r_f) + a(\tilde{r} - r_f)\right)(\tilde{r} - r_f)^2\right] < 0$, that is, $W'(a)$ is everywhere decreasing
  - It follows that $\hat{a}$ will be positive $\Leftrightarrow W'(0) > 0$
- Since $u' > 0$ this implies that $\hat{a} > 0 \Leftrightarrow E[\tilde{r} - r_f] > 0$
  - The other assertion follows similarly



Portfolio as wealth changes

- <u>Theorem (Arrow, 1971):</u>
  Let $\hat{a} = \hat{a}(Y_0)$ be the solution to max-problem above; then:
  i. $\frac{\partial R_A}{\partial Y} < 0$ (DARA) $\Rightarrow \frac{\partial \hat{a}}{\partial Y_0} > 0$
  ii. $\frac{\partial R_A}{\partial Y} = 0$ (CARA) $\Rightarrow \frac{\partial \hat{a}}{\partial Y_0} = 0$
  iii. $\frac{\partial R_A}{\partial Y} > 0$ (IARA) $\Rightarrow \frac{\partial \hat{a}}{\partial Y_0} < 0$

Portfolio as wealth changes

- Theorem (Arrow 1971): If, for all wealth levels Y,

i. $\frac{\partial R_R}{\partial Y} = 0$ (CRRA) $\Rightarrow \eta = 1$

ii. $\frac{\partial R_R}{\partial Y} < 0$ (DRRA) $\Rightarrow \eta > 1$

iii. $\frac{\partial R_R}{\partial Y} > 0$ (IRRA) $\Rightarrow \eta < 1$

where $\eta = \frac{da/a}{dY/Y}$

Log utility & Portfolio Allocation

$u(Y) = \ln Y$

$$E\left[\frac{(\tilde{r} - r_f)}{Y_0(1 + r_f) + a(\tilde{r} - r_f)}\right] = 0$$

$$\frac{a}{Y_0} = \frac{\left((1 + r_f)[E[\tilde{r}] - r_f]\right)}{-(r_1 - r_f)(r_2 - r_f)} > 0$$

2 states, where $r_2 > r_f > r_1$

Constant fraction of wealth is invested in risky asset!
Homework: show that this result holds for

- any CRRA utility function
- any distribution of r

Risk aversion and Portfolio Allocation

- Theorem (Cass and Stiglitz,1970): Let the vector $\begin{bmatrix} \hat{a}_1(Y_0) \\ \vdots \\ \hat{a}_J(Y_0) \end{bmatrix}$ denote the amount optimally invested in the J risky assets if the wealth level is Y_0.

  Then $\begin{bmatrix} \hat{a}_1(Y_0) \\ \vdots \\ \hat{a}_J(Y_0) \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_J \end{bmatrix} f(Y_0)$ if and only if either

  i. $u'(Y_0) = (BY_0 + C)^{\Delta}$ or
  ii. $u'(Y_0) = \xi e^{-\rho Y_0}$

- In words, it is sufficient to offer a **mutual fund**.

LRT/HARA-utility functions

- Linear Risk Tolerance/hyperbolic absolute risk aversion

$$-\frac{u''(c)}{u'(c)} = \frac{1}{A+Bc}$$

- Special Cases
  - $B=0, A>0$ CARA $\qquad u(c)=\frac{1}{B-1}(A+Bc)^{\frac{B-1}{B}}$
  - $B\neq 0, \neq 1$ Generalized Power
    - $B=1$ Log utility $\qquad u(c)=\ln[A+Bc]$
    - $B=-1$ Quadratic Utility $\qquad u(c)=-(A-c)^2$
    - $B\neq 1, A=0$ CRRA Utility function $\qquad u(c)=\frac{1}{B-1}(Bc)^{\frac{B-1}{B}}$

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6. Prudence coefficient and precautionary savings [DD5]
7. Mean-variance preferences [L4.6]

Digression: Subjective EU Theory

- Derive perceived probability from preferences!
  - Set S of prizes/consequences
  - Set Z of states
  - Set of functions $f(s) \in Z$, called acts (consumption plans)
- Seven SAVAGE Axioms
  - Goes beyond scope of this course.

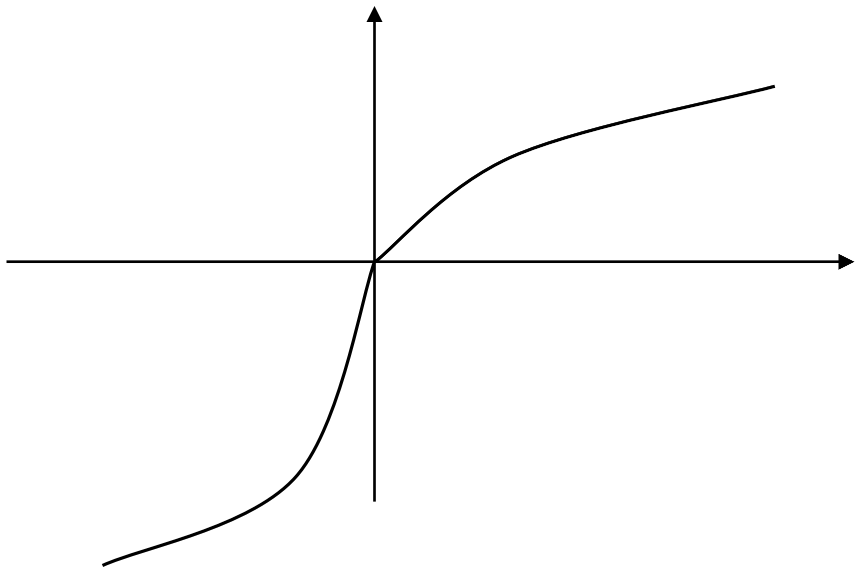
Digression: Ellsberg Paradox

- 10 balls in an urn
  Lottery 1: win $100 if you draw a red ball
  Lottery 2: win $100 if you draw a blue ball
- Uncertainty: Probability distribution is not known
- Risk: Probability distribution is known
  (5 balls are red, 5 balls are blue)

- Individuals are *"uncertainty/ambiguity averse"*
  (non-additive probability approach)

Digression: Prospect Theory

- Value function (over gains and losses)



- Overweight low probability events
- Experimental evidence

Overview: Risk Preferences

1. State-by-state dominance
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Introducing Savings

- Introduce savings decision: Consumption at $t = 0$ and $t = 1$
- *Asset structure 1:*
  - risk free bond R^f
  - NO risky asset with random return
- Increase R^f:
  - **Substitution effect:** shift consumption from $t = 0$ to $t = 1$ $\Rightarrow$ save more
  - **Income effect:** agent is “effectively richer” and wants to consume some of the additional richness at $t = 0$ $\Rightarrow$ save less
  - For log-utility ($\gamma = 1$) both effects cancel each other

Savings: Euler Equation

for CRRA: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$

- $\max_{c_0, c_1} u(c_0) + \delta u(c_1)$
  - s.t. $c_1 = R^f(e_0 - c_0) + e_1$
- $\max_{c_0} u(c_0) + \delta u(R^f e_0 + e_1)$
- FOC: $1 = \delta \frac{u'(c_1)}{u'(c_0)} R^f$
  - $\mathbb{r}^f \approx \ln R^f = -\ln\left(\frac{u'(c_1)}{u'(c_0)}\right) - ln\delta$

$1 = \delta \left(\frac{c_1}{c_0}\right)^{-\gamma} R^f$

for log: $u(c) = \ln c \;\&\; e_1 = 0$

$c_0 = \frac{1}{\delta(\delta+1)}[e_0 + \frac{1}{R} e_1]$

$c_1 = \left(1 - \frac{1}{\delta(\delta+1)}\right)[Re_0 + e\_1]$

for $e_1 = 0$ saving does not depend on (risk of) R^f: $1 = \delta\left(\frac{c_0}{R^f(e_0 - c_0)}\right) R^f$

Intertemporal Elasticity of Substitution

- $IES := \frac{\partial \ln\left(\frac{c_1}{c_0}\right)}{\partial r} = -\frac{\partial \ln\left(\frac{c_1}{c_0}\right)}{\partial \ln\left(\frac{u'(c_1)}{u'(c_0)}\right)}$

- For CRRA $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ $IES = \frac{1}{\gamma}$

Investment Risk

- Savings decision: Consumption at $t = 0$ and $t = 1$
- No endowment risk at $t = 1$
- Asset structure 2: (no portfolio choice yet)
  - Single risky asset only
  - No risk-free asset

- <u>Theorem (Rothschild and Stiglitz, 1971):</u>
For $R^B = R^A + \varepsilon$, where $\mathrm{E}[\varepsilon] = 0$ and $\varepsilon \perp R^A$, then respective savings s^A , s^B out of initial wealth level W_0 are
  - If $\frac{\partial R_R}{\partial W_0} \leq 0$ and $R_R > 1$, then $s^A < s^B$.
  - If $\frac{\partial R_R}{\partial W_0} \geq 0$ and $R_R < 1$, then $s^A > s^B$.

Investment Risk with Portfolio and Savings Decision

- Savings decision: Consumption at $t = 0$ and $t = 1$
- No endowment risk at $t = 1$, $e_1 = 0$
- Asset structure 3: portfolio shares α^j

- $\max_{c_0, c_1, \boldsymbol{\alpha_0}} u(c_0) + \delta E_0[u(c_1)]$
  s.t. $W_1 = \sum \alpha_0^j R_1^j (W_0 - c_0)$

$$u'(c_0) = E_0\left[\delta u'(c_1) R^j\right] \forall j$$

Investment Risk: Excess Return

$$1 = E_0\left[\delta \frac{u(c_1)}{u(c_0)} R^j\right] \ \forall j$$

- For CRRA $$1 = \delta E_0\left[\left(\frac{c_1}{c_0}\right)^{-\gamma} R^j\right]$$
- In "log-notation": $\mathbb{c}_t \equiv \log c_t$, $\mathbb{r}_t^j \equiv \log R_t^j$
$$1 = \delta E_0\left[e^{-\gamma(\mathbb{c}_1 - \mathbb{c}_0) + \mathbb{r}^j}\right]$$
- Assume $\mathbb{c}_t, \mathbb{r}_t^j \sim \mathcal{N}$
$$1 = \delta[e^{-\gamma E_0[\Delta \mathbb{c}_1] + E_0[\mathbb{r}^j] + \frac{1}{2} Var_0[-\gamma \Delta \mathbb{c}_1 + \mathbb{r}^j]}]$$
$$0 = \ln \delta - \gamma E_0[\Delta \mathbb{c}_1] + E_0[\mathbb{r}^j] + \frac{\gamma^2}{2} Var_0[\Delta \mathbb{c}_1] + \frac{1}{2} Var_0[\mathbb{r}^j] - \gamma Cov_0[\Delta \mathbb{c}_1, \mathbb{r}^j]$$
- For risk free asset:
$$\mathbb{r}^f = -\ln \delta + \gamma E_0[\Delta \mathbb{c}_1] - \frac{\gamma^2}{2} Var_0[\Delta \mathbb{c}_1]$$
- Excess return of any asset:
$$E_0[\mathbb{r}^j] + \frac{1}{2} Var_0[\mathbb{r}^j] - \mathbb{r}^f = \gamma Cov_0[\Delta \mathbb{c}_1, \mathbb{r}^j]$$

Investment Risk: Portfolio Shares

- Excess return

$$E_0[\mathbb{r}^j] + \frac{1}{2} Var_0[\mathbb{r}^j] - \mathbb{r}^f = \gamma Cov_0[\Delta \mathbb{c}_1, \mathbb{r}^j]$$

- If consumption growth $\Delta \mathbb{c}_1 = \Delta \mathbb{w}_1$ wealth growth
- $Cov_0[\Delta \mathbb{w}_1, \mathbb{r}^j] = Cov_0[\alpha_0^j \mathbb{r}^j, \mathbb{r}^j] = \alpha_0^j Var_0[\mathbb{r}^j]$
- Hence, optimal portfolio share

$$\alpha_0^j = \frac{E_0[\mathbb{r}^j] + \frac{1}{2} Var_0[\mathbb{r}^j] - \mathbb{r}^f}{\gamma Var_0[\mathbb{r}^j]}$$

Making $\Delta \mathbb{w}_1$ Linear in $\mathbb{c}_0 - \mathbb{w}_0$

- $W_1 = \sum \alpha_0^j R_1^j (W_0 - c_0)$ recall $e_1 = 0$
- $\frac{W_1}{W_0} = \sum \alpha_0^j R_1^j (1 - \frac{c_0}{W_0})$ let $R_1^p = \sum \alpha_0^j R_1^j$
- In "log-notation": $\Delta \mathbb{w}_1 = \mathbb{r}_1^j + \underbrace{\log(1 - e^{\mathbb{c}_0 - \mathbb{w}_0})}_{\text{nonlinear}}$
- Linearize using Taylor expansion around $\overline{\mathbb{c} - \mathbb{w}}$
- $\Delta \mathbb{w}_1 = \mathbb{r}_1^j + k + \left(1 - \frac{1}{\rho}\right)(\mathbb{c}_0 - \mathbb{w}_0)$
  - Where $k \equiv \log \rho + (1 - \rho) \log \frac{1-\rho}{\rho}$, $\rho = 1 - e^{\overline{\mathbb{c} - \mathbb{w}}}$

Hint: in continuous time this approximation is precise

Endowment Risk: Prudence and Pre-cautionary Savings

- Savings decision
  Consumption at $t = 0$ and $t = 1$
- *Asset structure 2:*
  - No investment risk: riskfree bond
  - Endowment at $t = 1$ is random (background risk)

- **2 effects:** Tomorrow consumption is more volatile
  - consume more today, since it's not risky
  - save more for precautionary reasons

Prudence and Pre-cautionary Savings

- Risk aversion is about the willingness to insure …
- … but not about its comparative statics.
- How does the behavior of an agent change when we marginally increase his exposure to risk?
- An old hypothesis (J.M. Keynes) is that
  - people save more when they face greater uncertainty
  - *precautionary saving*
- Two forms:
  - Shape of utility function u'''
  - Borrowing constraint $a_t \geq -b$

Precautionary Savings 1: Prudence

- Utility maximization $u(c_0) + \delta u E_0[u(c_1)]$
  - Budget constraint: $c_1 = e_1 + (1+r)(e_0 - c_0)$
  - Standard Euler equation: $u'(c_t) = \delta(1+r)E_t[u'(c_{t+1})]$
- If $u''' > 0$, then Jensen's inequality implies:
  - $\frac{1}{\delta(1+r)} = \frac{E_t[u'(c_{t+1})]}{u'(c_t)} > \frac{u'(E_t[c_{t+1}])}{u'(c_t)}$
  - Increase variance of e_1 (mean preserving spread)
  - Numerator $E_t[u'(c_{t+1})]$ increases with variance of c_{t+1}
  - For equality to hold, denominator has to increase
    c_t has to decrease,
    i.e. savings has to increase *precautionary savings*
- Prudence refers to curvature of u', i.e. $P = -\frac{u'''}{u''}$

Precautionary Savings 1: Prudence

- Does not directly follow from risk aversion, involves u'''
  - Leland (1968)
- Kimball (1990) defines **absolute prudence** as

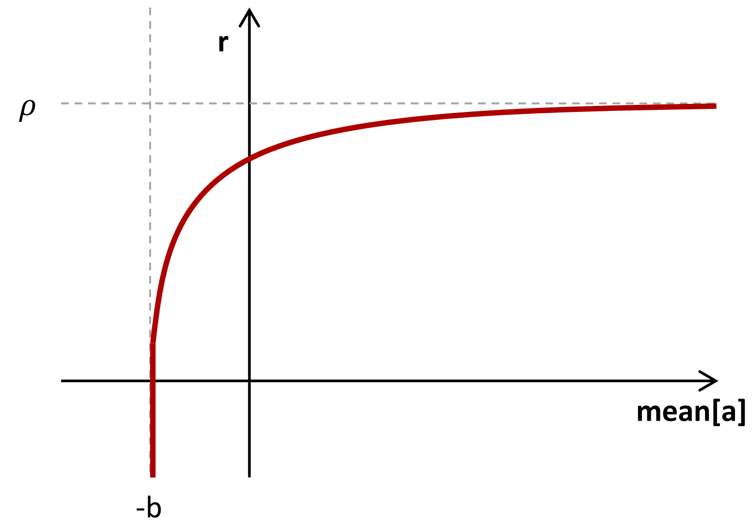
$$P(c) := -\frac{u'''(c)}{u''(c)}$$

- Precautionary saving if any only if prudent.
  - important for comparative statics of interest rates.
- DARA ⇒ Prudence

$$\frac{\partial\left(-\frac{u''}{u'}\right)}{\partial c} < 0, \qquad -\frac{u'''}{u''} > -\frac{u''}{u'}$$

Precautionary Savings 2: Future Borrowing Constraint

- Agent might be concerned that he faces *borrowing constraints* in some state in the future
- agents engage in precautionary savings (self-insurance)
- In Bewley (1977) idiosyncratic income shocks, mean asset holdings mean$[a]$ (across individuals) result from individual optimization



Precautionary Savings 1: Prudence

- *Asset structure 3:*
  - No risk free bond
  - One risky asset with random gross return R

- Theorem (Rothschild and Stiglitz,1971) : Let $\tilde{R}_A, \tilde{R}_B$ be two return distributions with identical means such that $\tilde{R}_B = \tilde{R}_A + e$, where e is white noise, and let s_A, s_B be the savings out of Y_0 corresponding to the return distributions $\tilde{R}_A, \tilde{R}_B$ respectively.
  - If $R_R'(Y) \leq 0$ and $R_R(Y) > 1$, then $s_A < s_B$
  - If $R_R'(Y) \geq 0$ and $R_R(Y) < 1$, then $s_A > s_B$

Precautionary Savings 1: Prudence

$$P(c) = -\frac{u'''(c)}{u''(c)}$$

$$P(c)c = -\frac{cu'''(c)}{u''(c)}$$

- Theorem: Let $\tilde{R}_A, \tilde{R}_B$ be two return distributions such that $\tilde{R}_A$ SSD $\tilde{R}_B$, let s_A and s_B be, respectively, the savings out of Y_0. Then,
  - $s_A \geq s_B \Leftrightarrow cP(c) \leq 2$ and conversely,
  - $s_A < s_B \Leftrightarrow cP(c) > 2$

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Mean-variance Preferences

- Early research (e.g. Markowitz and Sharpe) simply used mean and variance of return
- Mean-variance utility often easier than vNM utility function
- … but is it compatible with vNM theory?
- The answer is yes … approximately … under some conditions.

Mean-Variance: quadratic utility

Suppose utility is quadratic, $u(c) = ac - bc^2$

Expected utility is then

$$E[u(c)] = aE[c] - bE[c^2]$$
$$= aE[c] - b(E[c]^2 + \mathrm{var}[c])$$

Thus, expected utility is a function of the mean $E[c]$, and the variance $\mathrm{var}[c]$, only.

Mean-Variance: joint normals

- Suppose all lotteries in the domain have normally distributed prized. (independence is not needed).
  - This requires an infinite state space.
- Any linear combination of jointly normals is also normal.
- The normal distribution is completely described by its first two moments.
- Hence, expected utility can be expressed as a function of just these two numbers as well.

Mean-Variance: small risks

- Let $f: \mathbb{R} \to \mathbb{R}$ be a smooth function. The Taylor approximation is

$$f(x)$$
$$\approx f(x_0) + f'(x_0)\frac{(x-x_0)^1}{1!}$$
$$+ f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots$$

- Use the Taylor approximation for $E[u(x)]$

Mean-Variance: small risks

- Since $E[u(w+x)] = u(c^{CE})$, this simplifies to $w - c_{CE} \approx R_A(w)\frac{\text{var}(x)}{2}$
  - $w - c_{CE}$ is the risk premium
  - We see here that the risk premium is approximately a linear function of the variance of the additive risk, with the slope of the effect equal to half the coefficient of absolute risk.

Mean-Variance: small risks

- Same exercise can be done with a multiplicative risk.
- Let $y = gw$, where g is a positive random variable with unit mean.
- Doing the same steps as before leads to

$$1 - \kappa \approx R_R(w) \frac{\mathrm{var}[g]}{2}$$

  - where κ is the certainty equivalent growth rate, $u(\kappa w) = E[u(gw)]$.
  - The coefficient of *relative* risk aversion is relevant for multiplicative risk, absolute risk aversion for additive risk.